Borda-Optimal Linear Taxation of Labour Income*

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Abstract

I study theoretically and numerically Borda-optimal (BO), i.e., optimal based on the Borda count as the normative criterion, linear taxation of labour income. On the theoretical side, I derive a first-order condition that any BO linear tax schedule must satisfy. Based on this condition, I derive two theoretical results. First, an increase in inequality has an ambiguous effect on the progressivity of the BO linear tax schedule. Second, in the special case of unitary elasticity of labour supply, the BO linear tax schedule is more progressive than the median-type's optimal feasible linear tax schedule and, hence, than the majority-rule linear tax schedule. This result holds up in numerical analysis based on different elasticities of labour supply.

Keywords: Borda count, preference aggregation, labour income taxation

JEL classification: D71, H21, H24

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1 Introduction

I study theoretically and numerically Borda-optimal (BO), i.e., optimal based on the Borda count as the normative criterion, linear taxation of labour income. I do so in the context of a static model with quasilinear preferences, a constant elasticity of labour supply (σ), and productivities (types) that are private information. Although the model is highly stylised and the linearity of tax schedules is restrictive, the set-up is rich enough for studying (in an admittedly crude way) the optimal progressivity of labour-income taxation.

On the theoretical side, I derive a first-order condition that any BO linear tax schedule must satisfy. Based on this condition, I derive two theoretical results. First, an increase in inequality has an ambiguous effect on the progressivity of the BO linear tax schedule. Second, in the special case of $\sigma = 1$, the BO linear tax schedule is more progressive (strictly so under an additional, plausible assumption) than the median-type's optimal feasible linear tax schedule and, hence, than the majority-rule linear tax schedule.

I complement the theory with numerical analysis of calibrations of the model that are tailored to the United States and assume different values of σ. This yields two findings. First, for each value of σ considered, the BO linear tax schedule is strictly more progressive than the median-type's optimal feasible linear tax schedule. Second, the progressivity of the BO linear tax schedule decreases sharply as σ increases.

1.1 Literature on Optimal Taxation of Labour Income

There exists a literature on optimal linear taxation of labour income based on utilitarian and Rawlsian normative criteria (e.g., Sheshinski (1972), Helpman and Sadka (1978), Hellwig (1986)). The most relevant for the current paper analysis is given in Helpman and Sadka (1978). In particular, the authors find that a mean-preserving spread of the distribution of productivities may increase or leave unaffected the progressivity of the Rawlsian-optimal linear tax schedule. They also conjecture that such a spread could have an effect of either sign on the progressivity of the utilitarian-optimal linear tax schedule. Such ambiguous effects of increases in inequality are in line with my results.

There is also a literature on linear taxation of labour income based on majority rule (e.g., Romer (1975), Roberts (1977), Meltzer and Richard (1981)). The most relevant for the current paper finding in this literature is that, under certain assumptions, the progressivity of the majority-rule linear tax schedule increases as inequality (measured by the ratio between mean and median income) increases (Meltzer and Richard (1981)).

There exists also a large literature on optimal nonlinear taxation of labour income. The bulk of this literature is based on a utilitarian or Rawlsian criterion (e.g., Mirrlees (1971), Diamond (1998), and Saez (2001)). The only paper that is based on the Borda count is, to the best of my knowledge, Ivanov (2022a).[1] This literature largely emphasises issues such as the shape of the optimal marginal-rate schedule, the sign of the optimal marginal rates, and the magnitude of the optimal marginal rate at high incomes. These issues are moot for the purposes of the current paper.

1.2 The Borda Count vs. Other Normative Criteria

The Borda count has several important advantages as a normative criterion. First, it has been characterised in terms of normatively appealing axioms.[2] For the purposes

[1] Ivanov (2022a) considers BO piecewise linear tax schedules. All the findings in that paper are obtained based on numerical computations.

[2] Young (1974) and Maskin (2021) characterise the Borda count under the assumption that individuals' preferences between alternatives are strict. For the case in which individuals are allowed to have weak preferences, (i) Young notes that his characterisation theorem can be proved in much the same way and (ii) Ivanov (2022b) shows that the Borda count satisfies (extensions to the case of weak preferences of) Maskin's axioms as well as an additional normatively appealing axiom. In the current paper, individuals can exhibit indifference between tax schedules. In addition, because the

of the current paper, the key distinction between the Borda count and majority rule is that majority rule satisfies Arrow's independence of irrelevant alternatives (IIA) whereas the Borda count only satisfies a weakening of IIA, called modified IIA, proposed in Maskin (2021). Thus, whether the Borda count or majority rule is deemed normatively preferable comes down to one's views on the normative merits of modified IIA vs. IIA.[3] Maskin (2021) argues in favour of the former and Pearce (2021) argues forcefully against the latter.

Second, preference aggregation based on some "reasonable" method such as the Borda count (or majority rule) seems central to the idea of democracy. In contrast, the utilitarian and Rawlsian criteria seem disconnected from this idea. This is awkward given the broad consensus in many countries that public policy should be determined through a democratic process.

Third, unlike the utilitarian and Rawlsian criteria, the Borda count can be implemented without going beyond ordinal utility. The flip side of this is that, although the Borda count exhibits some sensitivity to the intensity of preferences between any two alternatives by taking into account the number of alternatives ranked in between by each individual, a policy-maker may wish to be more sensitive to preference intensities (e.g., based on introspection or individuals' verbal reports).[4]

In any case, it is safe to say that the Borda count is a main contender as far as normative criteria go. However, its implications for optimal public policy are relatively unexplored. The current paper aims to help fill this gap.

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existing literature on the Borda count assumes finitely many alternatives whereas there are infinitely many linear tax schedules, I propose a natural modification of the Borda count. This modification has not been axiomatised.

[3] An important limitation of majority rule is that in many contexts a Condorcet winner may not exist. In the current paper, this point is moot because a Condorcet winner does always exist.

[4] A utilitarian or Rawlsian criterion may be better able to accommodate such additional sensitivity. However, it is hard for the researcher writing the optimal-policy paper to know what utility functions to use in the social welfare function given that the appropriate utility functions would be based on each policy-maker's subjective judgements and are also likely to differ across policy-makers.

2 Set-Up

2.1 Preferences and Productivities

Individuals have preferences over consumption $c \geq 0$ and labour $l \geq 0$ represented by the utility function $c - \frac{\sigma}{1+\sigma} l^{\frac{1+\sigma}{\sigma}}$, where $\sigma > 0$ is the (Hicksian and Marshallian) elasticity of labour supply. Each individual has a productivity (or type) $w \geq 0$ which is her private information. When type w puts in labour l, she earns pre-tax income wl. There is a unit mass of individuals whose types are distributed according to the cumulative density function (CDF) F.

2.2 Tax Schedules

I restrict attention to linear tax schedules. Given the slope (i.e., the tax rate) of a tax schedule, its intercept (i.e., the universal basic income (UBI)) is pinned down via the government budget constraint. In particular, given tax rate t, individuals receive a per-capita UBI equal to $\int_0^\infty twl(w)dF(w) - R$, where $l(w) = \begin{cases} (1-t)^\sigma w^\sigma & \text{if } t \leq 1 \\ 0 & \text{if } t > 1 \end{cases}$ is type w's optimal labour supply and $R > 0$ is the exogenously given government consumption. Letting I denote $\int_0^\infty w^{1+\sigma} dF(w)$, the UBI can be expressed as $\begin{cases} t(1-t)^\sigma I - R & \text{if } t \leq 1 \\ -R & \text{if } t > 1 \end{cases}$.[5] A tax rate is feasible if the associated UBI is nonnegative so that even type $w = 0$ obtains nonnegative consumption.[6] Given that $t(1-t)^\sigma I - R$ is an inverted-U-shaped function of t on $(-\infty, 1]$ and assuming $t(1-t)^\sigma I > R$ at the revenue-maximising tax rate $t = 1/(1+\sigma)$, the set of feasible tax rates is of the form $[\underline{t}, \overline{t}]$, where $0 < \underline{t} < \overline{t} < 1$.

[5] I assume $I < \infty$ (so that the UBI is finite).

[6] This definition of feasibility implicitly assumes that $w = 0$ is in the support of F.

2.3 Indirect Utility over Tax Rates

Given $t \leq 1$, type w's indirect utility is[7]

$$v(t,w) = \underbrace{(1-t)wl(w)}_{\text{after-tax income}} + \underbrace{t(1-t)^{\sigma} I - R}_{\text{UBI}} - \underbrace{\frac{\sigma}{1+\sigma} l(w)^{\frac{1+\sigma}{\sigma}}}_{\text{disutility from labour}} = \tag{1}$$

$$\frac{1}{1+\sigma}(1-t)^{\sigma}\left(w^{1+\sigma} + \left((1+\sigma)I - w^{1+\sigma}\right)t\right) - R. \tag{2}$$

Lemma 1 establishes key properties of v and each type's optimal feasible tax rate.[8]

Lemma 1

1) For any $t \in [\underline{t}, \overline{t})$ and any type w, $v(t,w) > v(\overline{t}, w)$.

2) For each type w, there is a unique optimal feasible tax rate

$$t^*(w) = \begin{cases} \frac{I - w^{1+\sigma}}{(1+\sigma)I - w^{1+\sigma}} & \text{if } w \leq \left(\frac{1-(1+\sigma)\underline{t}}{1-\underline{t}} I\right)^{1/(1+\sigma)} \\ \underline{t} & \text{otherwise} \end{cases}. \tag{3}$$

3) $t^*(\cdot)$ is continuous and strictly decreasing on $[0, \left(\frac{1-(1+\sigma)\underline{t}}{1-\underline{t}} I\right)^{1/(1+\sigma)}]$.

4) For each type w, $v(\cdot, w)$ is strictly increasing on $[\underline{t}, t^*(w)]$ and strictly decreasing on $[t^*(w), \overline{t}]$.

For future reference, let v_t denote the derivative of v with respect to its first argument. Also, given $t \leq 1$ and $w \geq 0$, define $\tau(t,w)$ implicitly by the conditions (i) $v(\tau(t,w), w) = v(t,w)$ and (ii) $\tau(t,w) \neq t$ whenever more than one value of $\tilde{t}$ solves $v(\tilde{t}, w) = v(t,w)$.[9]

[7]This specification of indirect utility implicitly assumes that each individual's preference over tax schedules is selfish. This assumption is discussed further in the appendix in Ivanov (2022a).

[8]All proofs that are not given in the main text can be found in the appendix.

[9]It is straightforward to verify based on (2) that at most two values of $\tilde{t}$ solve $v(\tilde{t}, w) = v(t,w)$.

3 The Borda Count

3.1 Definition

With a finite number of alternatives, the points that an alternative, x, obtains from a given individual in the computation of the Borda count equal the number of alternatives the individual ranks strictly below x minus the number of alternatives the individual ranks strictly above x.[10] When the set of alternatives consists of all tax rates in $[\underline{t}, \overline{t}]$, the natural modification is to let the points that tax rate, t, obtains from an individual of type w equal $\beta(t,w) - \alpha(t,w)$, where $\beta(t,w)$ (respectively, $\alpha(t,w)$) denotes the Lebesgue measure of tax rates in $[\underline{t}, \overline{t}]$ that type w ranks strictly below (respectively, strictly above) t. Moreover, given part 4) of Lemma 1, individuals can never be indifferent between a positive measure of tax rates in $[\underline{t}, \overline{t}]$ so that $\beta(t,w) - \alpha(t,w) = \overline{t} - \underline{t} - 2\alpha(t,w)$. Ignoring irrelevant constants and integrating over types leads to the following definition of the Borda Count for $t \in [\underline{t}, \overline{t}]$ given F:

$$B(t,F) = -\int_0^\infty \alpha(t,w) dF(w). \tag{4}$$

A BO tax rate (given F) is one the maximises $B(\cdot, F)$ on $[\underline{t}, \overline{t}]$.[11]

Note that tax rates above the revenue-maximising rate, $\frac{1}{1+\sigma}$, are Pareto dominated and can never maximise $B(\cdot, F)$. Hence, I will restrict attention to $B(\cdot, F)$ on $[\underline{t}, \frac{1}{1+\sigma}]$. The next section provides a useful way of rewriting $B(\cdot, F)$ on $[\underline{t}, \frac{1}{1+\sigma}]$.

[10]This is the case in the generalisation of the Borda count for the case in which (like in the current paper) individuals can exhibit indifference between alternatives.

[11]I treat $[\underline{t}, \overline{t}]$ as the set of alternatives. This is unusual because $\underline{t}$ and $\overline{t}$ depend on the profile of individuals' preferences over consumption and labour whereas in social choice theory the set of alternatives is specified independently of preferences. An equivalent approach that avoids this problem is to specify the set of alternatives as $(0,1)$ and to assume that individuals rank all tax rates in $(0,1) \setminus [\underline{t}, \overline{t}]$ as strictly worse (because of their infeasibility) than any $t \in [\underline{t}, \overline{t}]$. The justification for restricting the set of alternatives to $(0,1)$ is that (i) $t \in (-\infty, 0]$ cannot be feasible for any preference profile and (ii) $t \in [1, \infty)$ cannot be feasible for any preference profile in which each individual strictly prefers (c,l) to (c',l') whenever $c' \leq c$ and $l' > l = 0$.

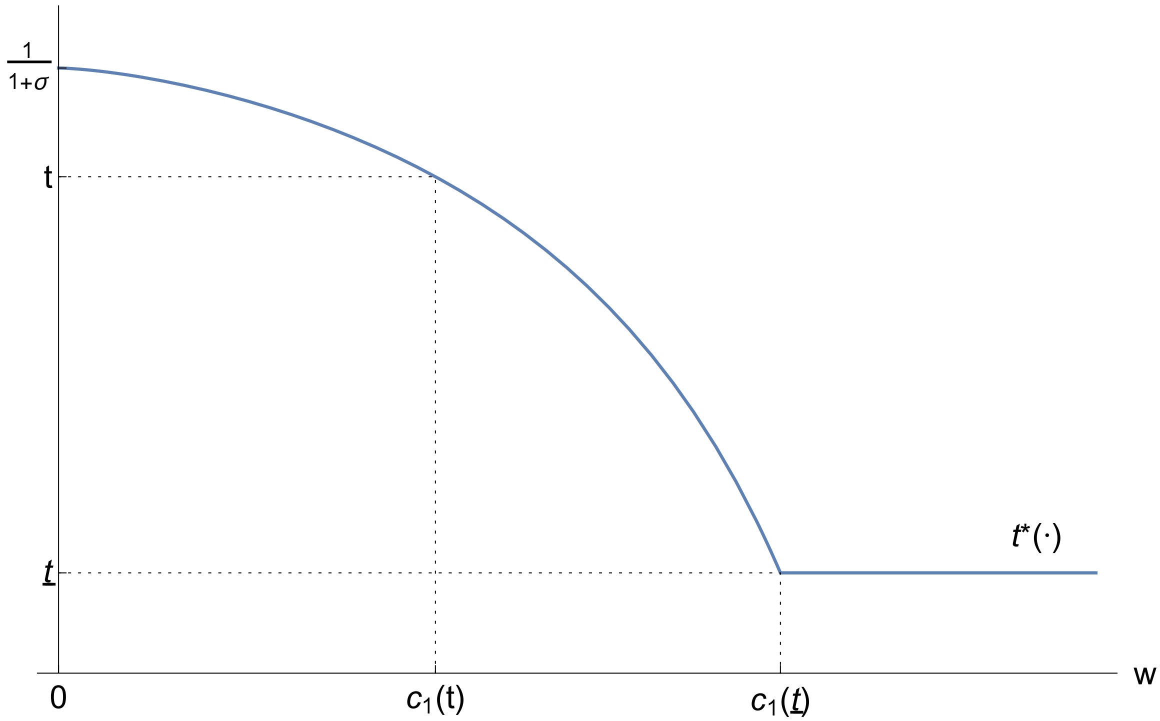

Figure 1: $t^*(\cdot)$ and $c_1(\cdot)$.

3.2 Rewriting the Borda Count

Let $c_1 : [\underline{t}, \frac{1}{1+\sigma}] \to \mathbb{R}$ be the inverse of $t^*(\cdot)$, when the domain of the latter is restricted to $[0, \left(\frac{1-(1+\sigma)\underline{t}}{1-\underline{t}} I\right)^{1/(1+\sigma)}]$. Figure 1 illustrates how $c_1(t)$ is determined. Note that $c_1(t)$ is a threshold type that separates types based on how their optimal feasible tax rates compare to t.

Also, define $c_2 : [\underline{t}, \frac{1}{1+\sigma}] \to \mathbb{R}$ as follows: $c_2(\underline{t}) = c_1(\underline{t})$ and, for $t \in (\underline{t}, \frac{1}{1+\sigma}]$, $c_2(t)$ is the (as can easily be shown) unique $w \geq 0$ for which $v(t, w) = v(\underline{t}, w)$.

Lemma 2

1) For any $t \in (\underline{t}, \frac{1}{1+\sigma}]$, $v(\underline{t}, w) \leq v(t, w)$ if and only if $w \leq c_2(t)$.

2) For any $t \in (\underline{t}, \frac{1}{1+\sigma}]$, $c_2(t) > c_1(t)$.

3) For any $t \in [\underline{t}, \frac{1}{1+\sigma}]$, $F(c_2(t)) < 1$.

4) $\frac{v_t(t,w)}{v_t(\tau(t,w),w)} < 0$ *for any* $t \in [\underline{t}, \frac{1}{1+\sigma}]$ *and* $w \in [0, c_1(t)) \cup (c_1(t), c_2(t)]$.

Part 1) of Lemma 2 makes clear that, for $t \in (\underline{t}, \frac{1}{1+\sigma}]$, $c_2(t)$ is a threshold type that separates types based on how $v(\underline{t}, w)$ and $v(t, w)$ compare. Parts 2)-4) will be useful further below. Note that part 4) holds because, for t and w in the given ranges, $v(\cdot, w)$ has a peak and t and $\tau(t, w)$ lie on opposite sides of that peak.

Lemma 3 gives the functional form of $\alpha(t, w)$ based on $c_1(t)$, $c_2(t)$, and $\tau(t, w)$.

Lemma 3 *For any* $t \in [\underline{t}, \frac{1}{1+\sigma}]$,

$$\alpha(t,w) = \begin{cases} \tau(t,w) - t & \text{if } w \leq c_1(t) \\ t - \tau(t,w) & \text{if } c_1(t) < w \leq c_2(t) \\ t - \underline{t} & \text{if } w > c_2(t) \end{cases} . \tag{5}$$

Figure 2 depicts $\alpha(t, w)$ for $t > \underline{t}$ and $w \in \{w_1, w_2, w_3\}$, where $w_1 \leq c_1(t) < w_2 \leq c_2(t) < w_3$. The proof shows that, given Lemmas 1 and 2, the functions $v(\cdot, w_1)$, $v(\cdot, w_2)$, and $v(\cdot, w_3)$ are as drawn in the figure.[12]

Given expression (5) and part 2) of Lemma 2, the Borda count in (4) can be rewritten for any $t \in [\underline{t}, \frac{1}{1+\sigma}]$ as:

$$B(t,F) = -\int_0^{c_1(t)} (\tau(t,w) - t)dF(w) - \int_{c_1(t)}^{c_2(t)} (t - \tau(t,w))dF(w) - (t - \underline{t}) \int_{c_2(t)}^{\infty} dF(w) \tag{6}$$

3.3 Existence of a BO Tax Rate

Assumption 1 *F has an associated continuous probability density function (PDF), f.*

Lemma 4 *If Assumption 1 holds, a BO tax rate exists.*

[12] The corresponding figure for the case $t = \underline{t}$ does not add much insight. It is considered as part of the proof.

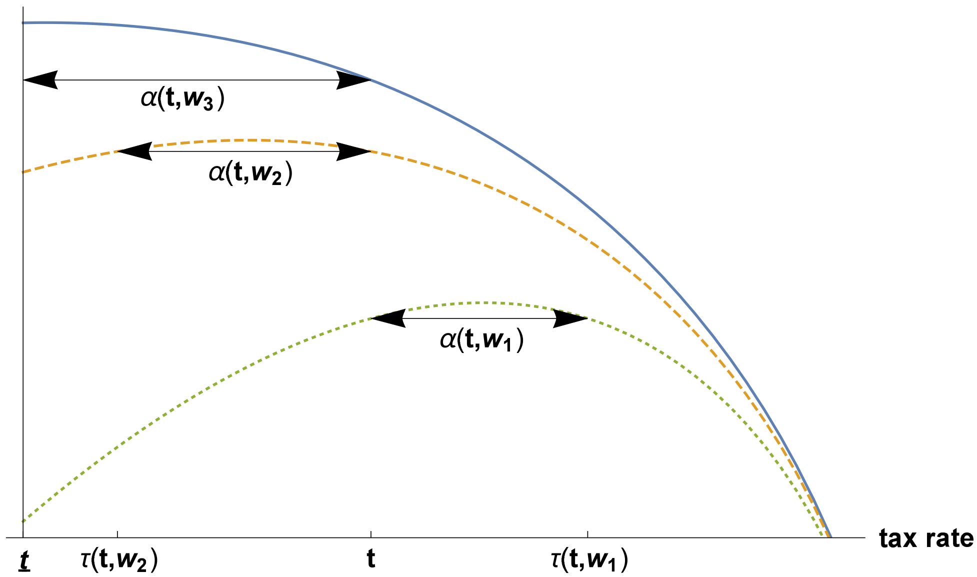

Figure 2: $\alpha(t, w_1)$, $\alpha(t, w_2)$, and $\alpha(t, w_3)$, where $t > \underline{t}$ and $w_1 \leq c_1(t) < w_2 \leq c_2(t) < w_3$. (If $w_1 = c_1(t)$, the peak of $v(\cdot, w_1)$ occurs at t so that $\tau(t, w_1) = t$ and $\alpha(t, w_1) = 0$.)

The proof of the lemma shows that $B(\cdot, F)$ in (6) is continuous on $[\underline{t}, \frac{1}{1+\sigma}]$ so that it must have a maximum on that interval.

3.4 First-Order Condition

Propositions 1 and 2 below will make the following "technical" assumption.

Assumption 2 *For any $w \in [0, c_1(\underline{t})]$, the derivative of $\tau(\cdot, \cdot)$ with respect to its first argument, τ_t, is continuous at $(t^*(w), w)$ if it exists at that point.*

Although I have not been able to prove this assumption, the proof of Proposition 1 shows that, at $(t^*(w), w)$ (where $w \in [0, c_1(\underline{t})]$), τ_t exists and is continuous in two directions in t-w space: (i) in the first argument and (ii) along the $c_1(\cdot)$ curve. This strongly suggests that, absent mathematical pathologies, Assumption 2 holds.

Proposition 1 *Suppose Assumptions 1 and 2 hold.*

1) Given $t \in [\underline{t}, \frac{1}{1+\sigma}]$*, the derivative of* $B(\cdot, F)$ *is*

$$B_t(t,F) = \int_0^{c_1(t)} \left(1 - \frac{v_t(t,w)}{v_t(\tau(t,w),w)}\right) f(w)dw - \int_{c_1(t)}^{c_2(t)} \left(1 - \frac{v_t(t,w)}{v_t(\tau(t,w),w)}\right) f(w)dw - \int_{c_2(t)}^{\infty} f(w)dw. \tag{7}$$

2) If t *is a BO tax rate, then*

$$B_t(t,F) \leq 0, \tag{8}$$

where the inequality holds with equality if $t > \underline{t}$*.*

The form of $B_t(t,F)$ in (7) is obtained by applying the Leibniz integral rule to $B(t,F)$ in (6) and using the implicit function theorem to express $\tau_t(t,w)$ as $\frac{v_t(t,w)}{v_t(\tau(t,w),w)}$. The role of Assumptions 1 and 2 is to ensure that the Leibniz integral rule can indeed be used. Expression (8) is a standard first-order condition.[13]

4 BO Tax Rate and Increases in Inequality

Consider a PDF over productivities, g, such that (i) g is continuous, (ii) $\int_0^\infty wg(w)dw = \int_0^\infty wf(w)dw$, and (iii) for some $0 \leq k_1 < k_2 \leq k_3 < k_4 \leq k_5 < k_6$, we have $g > f$ on $(k_1,k_2) \cup (k_5,k_6)$, $g < f$ on (k_3,k_4), and $g = f$ outside of $(k_1,k_2) \cup (k_3,k_4) \cup (k_5,k_6)$. Thus, g maintains the mean of f while increasing inequality by shifting probability mass away from the middle interval (k_3,k_4) and towards the outer intervals (k_1,k_2) and (k_5,k_6). Let $t^{\mathrm{BO}}(f)$ and $t^{\mathrm{BO}}(g)$ be the sets of BO tax rates if types are distributed according to f and g, respectively.

[13] The proof of Proposition 1 shows that $B_t(\frac{1}{1+\sigma}, F) < 0$. Thus, we need not be concerned that $t = \frac{1}{1+\sigma}$ is BO and $B_t(t,F) > 0$ at that corner.

Proposition 2 *Suppose that Assumptions 1 and 2 hold.*

1) *There exists g satisfying (i)-(iii) above such that* $t^{BO}(f) = t^{BO}(g)$.

2) *There exists g satisfying (i)-(iii) above such that, for any* $t_f \in t^{BO}(f)$ *and any* $t_g \in t^{BO}(g)$, $t_f \leq t_g$, *the inequality being strict if* $t_f > \underline{t}$.

3) *There exists g satisfying (i)-(iii) above such that, for any* $t_f \in t^{BO}(f)$ *and any* $t_g \in t^{BO}(g)$, $t_f \geq t_g$, *the inequality being strict if* $t_g > \underline{t}$.

The punchline of Proposition 2 is that an increase in inequality can have an ambiguous effect on the set of BO tax rates.

In the proof below, G denotes the CDF associated with g.

Proof of part 1):

Take $k_1 \geq c_2(t)$. In this case, relative to f, g just shifts around probability mass on $(c_2(t), \infty)$.[14] Thus, as is evident from (6), $B(t, F) = B(t, G)$ for all $t \in [\underline{t}, \frac{1}{1+\sigma}]$. Q.E.D.

Proof of part 2):

Take $k_2 \leq c_1(t) \leq k_3$ and $c_2(t) \leq k_5$. In this case, relative to f, g shifts probability mass from $(c_1(t), \infty)$ to $(0, c_1(t))$ and does not add probability mass to $(c_1(t), c_2(t))$.[15]

Given Theorems 2.8.4 and 2.8.5 in Topkis (1998), it suffices to show that $B_t(t, G)-$

[14] Given part 3) or Lemma 2, f puts positive mass on $(c_2(t), \infty)$. Hence, k_3 and k_4 can be chosen so that $g < f$ on (k_3, k_4).

[15] Given part 3) or Lemma 2, f puts positive mass on $(c_1(t), \infty)$. Hence, k_3 and k_4 can be chosen so that $g < f$ on (k_3, k_4).

$B_t(t, F) > 0$ for all $t \in [\underline{t}, \frac{1}{1+\sigma}]$. For any $t \in [\underline{t}, \frac{1}{1+\sigma}]$, we have

$$B_t(t,G) - B_t(t,F) =$$
$$\int_0^{c_1(t)} \left(1 - \frac{v_t(t,w)}{v_t(\tau(t,w),w)}\right)(g(w) - f(w))dw +$$
$$\int_{c_1(t)}^{c_2(t)} \left(1 - \frac{v_t(t,w)}{v_t(\tau(t,w),w)}\right)(f(w) - g(w))dw + \int_{c_2(t)}^{\infty} (f(w) - g(w))dw \geq$$
$$\int_{c_1(t)}^{c_2(t)} \left(1 - \frac{v_t(t,w)}{v_t(\tau(t,w),w)}\right)(f(w) - g(w))dw + \int_{c_2(t)}^{\infty} (f(w) - g(w))dw \geq$$
$$\int_{c_1(t)}^{\infty} (f(w) - g(w))dw > 0.$$

The first inequality holds because, for all $w \in (0, c_1(t))$, $1 - \frac{v_t(t,w)}{v_t(\tau(t,w),w)} > 1$ (by part 4) of Lemma 2) and $g(w) \geq f(w)$. The second inequality follows because, for all $w \in (c_1(t), c_2(t))$, $1 - \frac{v_t(t,w)}{v_t(\tau(t,w),w)} > 1$ (by part 4) of Lemma 2) and $f(w) \geq g(w)$. The last inequality follows because g puts more mass than f on $(0, c_1(t))$ and, hence, less mass than f on $(c_1(t), \infty)$. Q.E.D.

Proof of part 3):

Take $c_1(t) \leq k_1 < k_2 \leq c_2(t) \leq k_3$. In this case, relative to f, g shifts probability mass from $(c_2(t), \infty)$ to $(c_1(t), c_2(t))$.[16]

Given Theorems 2.8.4 and 2.8.5 in Topkis (1998), it suffices to show that $B_t(t, G)-$

[16] Given part 3) or Lemma 2, f puts positive mass on $(c_2(t), \infty)$. Hence, k_3 and k_4 can be chosen so that $g < f$ on (k_3, k_4).

$B_t(t, F) < 0$ for all $t \in [\underline{t}, \frac{1}{1+\sigma}]$. For any $t \in [\underline{t}, \frac{1}{1+\sigma}]$, we have

$$
\begin{aligned}
&B_t(t, G) - B_t(t, F) = \\
&\int_0^{c_1(t)} \left(1 - \frac{v_t(t, w)}{v_t(\tau(t, w), w)}\right) (g(w) - f(w))dw + \\
&\int_{c_1(t)}^{c_2(t)} \left(1 - \frac{v_t(t, w)}{v_t(\tau(t, w), w)}\right) (f(w) - g(w))dw + \int_{c_2(t)}^{\infty} (f(w) - g(w))dw = \\
&\int_{c_1(t)}^{c_2(t)} \left(1 - \frac{v_t(t, w)}{v_t(\tau(t, w), w)}\right) (f(w) - g(w))dw + \int_{c_2(t)}^{\infty} (f(w) - g(w))dw < \\
&\int_{c_1(t)}^{\infty} (f(w) - g(w))dw = 0
\end{aligned}
$$

The second equality holds because $f = g$ on $(0, c_1(t))$. The inequality follows because $1 - \frac{v_t(t,w)}{v_t(\tau(t,w),w)} > 1$ for all $w \in (c_1(t), c_2(t))$ (recall part 4) of Lemma 2), $f \leq g$ on $(c_1(t), c_2(t))$, and $f < g$ on $(k_1, k_2) \subseteq (c_1(t), c_2(t))$. The third equality holds because f and g put the same mass on $(c_1(t), \infty)$. Q.E.D.

5 Borda Count vs. Majority Rule when $\sigma = 1$

Let w_{50} denote the median type (which I assume is unique). Given that, by part 4) of Lemma 1, $v(\cdot, w)$ is single-peaked on $[\underline{t}, \overline{t}]$ for each w, the following result is not surprising.

Lemma 5 *For any $t \in [\underline{t}, \overline{t}]$ such that $t \neq t^*(w_{50})$, strictly more than half of the population strictly prefers $t^*(w_{50})$ to t.*

The following proposition compares, for the special case when $\sigma = 1$, the BO tax rate with $t^*(w_{50})$ and, hence, with the majority-rule (MR) tax rate.

Proposition 3 *Suppose that Assumption 1 holds and $\sigma = 1$. If t^{BO} is a BO tax rate, then $t^{BO} \geq t^*(w_{50})$. Moreover, the inequality is strict if $\int_0^{c_1(\underline{t})} f(w)dw > 1/3$.*

The condition $\int_0^{c_1(\underline{t})} f(w)dw > 1/3$ requires that more than one-third of individuals have an optimal feasible tax rate strictly above $\underline{t}$ (see Figure 1). This seems like the empirically relevant case.

Proof:

Suppose $\sigma = 1$. As shown in the completion of this proof in the appendix, (i) Assumption 2 holds and (ii) the $\frac{v_t(t,w)}{v_t(\tau(t,w),w)}$ terms under the integrals in (7) equal -1. As a result of (i), expression (7) for $B_t(t,F)$ and the first-order condition (8) apply. As a result of (ii), (8) simplifies to

$$\int_0^{c_1(t)} f(w)dw \leq \int_{c_1(t)}^{c_2(t)} f(w)dw + \frac{1}{2}\int_{c_2(t)}^{\infty} f(w)dw. \tag{9}$$

First, suppose that $t^*(w_{50}) > \underline{t}$ and consider any $t \in [\underline{t}, t^*(w_{50})]$. Based on Figure 1, it is clear that $c_1(t) \geq w_{50}$. Thus, the left-hand side of (9) is greater than or equal to 0.5 while, given that by part 3) of Lemma 2 there is a positive mass of types on $(c_2(t), \infty)$, the right-hand side of (9) is strictly below 0.5. Thus, t cannot be BO.

Next, suppose that $t^*(w_{50}) = \underline{t}$. Then, $t^{\mathrm{BO}} \geq t^*(w_{50})$ must hold. If we further assume $\int_0^{c_1(\underline{t})} f(w)dw > 1/3$, then, at $t = \underline{t}$, (i) the left hand-side of (9) is strictly above 1/3 while (ii) the right-hand side equals $\frac{1}{2}\int_{c_1(\underline{t})}^{\infty} f(w)dw = \frac{1}{2}\left(1 - \int_0^{c_1(\underline{t})} f(w)dw\right) <$ 1/3. Thus, $t^*(w_{50})$ cannot be BO. Q.E.D.

6 Numerical Analysis

$\sigma = 1$ is probably towards the high end of empirically plausible values. Thus, I next explore numerically how the (as it will turn out unique) BO linear tax schedule and the MR linear tax schedule compare under different values of σ. The numerical analysis will also shed some light on how the BO tax rate depends on σ.

6.1 Calibration

6.1.1 Elasticity of Labour Supply

There is considerable controversy regarding the responsiveness of labour supply to wages and taxes.[17] Therefore, in a model with quasilinear preferences and a constant elasticity of labour supply, Saez and Stantcheva (2018) perform their computations separately for $\sigma \in \{0.25, 0.5, 1\}$. I follow these authors and perform my computations for these three values of σ. In addition, because Proposition 3 applies to the case $\sigma = 1$, it makes sense to also perform the computations for some $\sigma > 1$. Thus, I also consider $\sigma = 1.5$.

6.1.2 Distribution of Types

The main idea for calibrating the distribution of types goes as follows. First, I assume that the actual labour-income tax schedule is linear with a 30 percent marginal tax rate. Given this tax schedule, type w's optimal pretax labour income is $y^*(w) = 0.7^\sigma w^{1+\sigma}$. Second, I back out the distribution of types based on $y^*(\cdot)$ and data from the World Inequality Database (WID) on the empirical distribution of pretax labour income for individuals over age 20 in the US in 2014.[18]

6.1.3 Government Consumption Per Capita

According to WID, US national income per individual over age 20 in 2014 was \$65,192.[19] According to Piketty, Saez, and Zucman (2018), total (i.e., federal, state, and local) government consumption in the US has been around 18 percent of national income since the end of World War II. Thus, I set $R = 65,192 \times 0.18 \approx 11,735$. This

[17] Keane (2011) and Saez et al. (2012) provide surveys of the literature.

[18] The details are in the appendix. The appendix also discusses some important aspects of the WID data

[19] All dollar amounts in the numerical analysis are in 2014 dollars.

	Tax Rate				UBI			
	$\sigma = 0.25$	$\sigma = 0.5$	$\sigma = 1$	$\sigma = 1.5$	$\sigma = 0.25$	$\sigma = 0.5$	$\sigma = 1$	$\sigma = 1.5$
MR	0.61	0.44	0.28	0.24	\$11,114	\$5,285	\$754	\$0
BO	0.67	0.52	0.37	0.29	\$12,259	\$6,905	\$2,670	\$1,087

Table 1: BO and MR tax rates and UBI.

calculation assumes that government consumption must be financed entirely from labour income taxation, which seems like the natural theoretical benchmark based on Atkinson and Stiglitz (1976).

6.2 Results

Table 1 presents the BO and MR tax rates as well as the associated UBIs. The table yields the following two findings.

Finding 1 *For each $\sigma \in \{0.25, 0.5, 1, 1.5\}$, the BO tax rate and BO UBI are higher than the MR tax rate and MR UBI, respectively.*

Finding 2 *The BO tax rate and BO UBI sharply decrease as σ increases on $\{0.25, 0.5, 1, 1.5\}$.*

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7 Appendix: Distribution of Types

7.1 Calibration

I assume that the actual labour-income tax schedule is a 30 percent flat tax. Given this tax schedule, type w's optimal pretax labour income is $y^*(w) = 0.7^\sigma w^{1+\sigma}$.

I use data from WID on pretax labour income for individuals over the age of 20 in the US in 2014.[20] In particular, I obtain from WID the data presented in Table 2.

Percentile	**Pretax labour income**
5	1264.5269
10	4906.4861
15	7233.2855
20	9610.6254
25	12139.6792

[20] WID defines pretax labour income as the sum of all pretax personal income flows accruing to the individual owners of labor as a production factor, before taking into account the operation of the tax/transfer system, but after taking into account the operation of the pension system. The base unit is the individual (rather than the household) but resources are split equally within couples.

30	14567.6519
35	17096.7977
40	20030.5452
45	22964.2909
50	26403.9035
55	30652.7167
60	35407.4916
65	40465.6912
70	46434.3576
75	52807.7141
80	60698.4899
85	71017.3259
90	85989.5812
91	90238.4864
92	95195.5111
93	101063.0963
94	108245.7491
95	117350.5085
96	129490.1877
97	148711.4384
98	182095.5655
99	261003.6644
99.1	277189.9954
99.2	295399.505
99.3	315632.3865
99.4	342946.6831

99.5	377342.5145
99.6	426912.9817
99.7	495704.6629
99.8	621148.3276
99.9	925652.6564
99.91	987362.7844
99.92	1062224.324
99.93	1153272.102
99.94	1264552.771
99.95	1416299.129
99.96	1638860.375
99.97	1962585.89
99.98	2508872.558
99.99	3864473.291
99.991	4117383.735
99.992	4420876.636
99.993	4805300.271
99.994	5260539.622
99.995	5887757.761
99.996	6717304.348
99.997	7981856.566
99.998	10318750.34
99.999	15579289.96

Table 2: Various percentiles of pretax labour income.

I augment this data in two ways.[21] First, I assume that the lowest income equals \$0.[22] Second, WID does not report the income of the highest earner. It does report that the 99.999$^{\text{th}}$ income percentile equals \$15,579,290 and the average income in the top 0.001 percent equals \$32,134,644. I impute an income to the highest earner by assuming that this income and the 99.999$^{\text{th}}$ income percentile are symmetrically situated around \$32,134,644. That is, I assume that the highest earner has an income of \$48,689,999. I make this assumption on simplicity grounds. Given that the top 0.001 of earners earned only 0.7 percent of all income, it is unlikely that this assumption is of much consequence.

Then, using $y^*(\cdot)$ and the augmented WID income data, I back out the various type percentiles (i.e., the 0$^{\text{th}}$ percentile, the 100$^{\text{th}}$ percentile, and all the percentiles listed in Table 2). E.g., given that the 5$^{\text{th}}$ income percentile equals 1264.5269, I infer that the 5$^{\text{th}}$ type percentile is $w_5 = y^{*-1}(1264.5269) = 1264.5269^{\frac{1}{1+\sigma}}/0.7^{\frac{\sigma}{1+\sigma}}$, where $y^{*-1}(\cdot)$ denotes the inverse of $y^*(\cdot)$.

Finally, equipped with the various type percentiles, I specify the cumulative density function, F, of the distribution of types through linear interpolation. E.g., denoting the p^{th} type percentile by w_p, I assume that, on $[w_{10}, w_{15}]$, $F(w) = 0.1 + \frac{0.15-0.1}{w_{15}-w_{10}}(w - w_{10})$.

7.2 Comments on the WID Data

A few comments regarding the WID data on pretax labour income are in order. First, this data is based on all individuals over age 20 and it counts income from public and private pensions as labour income. This is not ideal for the purpose of backing out

[21]For brevity, in the rest of this section I will write "income" although in fact I mean "pretax labour income"

[22]WID reports a negative 0$^{\text{th}}$ income percentile. (I believe this is largely due to the partial imputation of the losses of privately owned businesses to labour income.) However, this is not consistent with the assumption that productivities are nonnegative.

productivities because the relationship between pension income and productivity is probably different from the relationship between a working-age individual's labour income and productivity.

Second, income is split equally within couples, which forces us to treat spouses as having the same productivity. This seems preferable for the purposes of the current paper because it ensures that the same preference over tax schedules is imputed to both spouses.

Third, although using cross-sectional data on the distribution of annual income to back out productivities is common (e.g., see Saez (2001)), this probably leads us to exaggerate the dispersion in lifetime productivities. The latter are probably more relevant if we are concerned with the design of a long-term tax system.[23]

8 Appendix: Proofs

8.1 A Preliminary Lemma

I start by stating and proving a preliminary lemma that will be useful in the proofs of the results stated in the main body of the paper.[24]

Lemma 0

1) For any $w < ((1+\sigma)I)^{1/(1+\sigma)}$, (i) $v(\cdot, w)$ is strictly increasing on $(-\infty, \frac{I-w^{1+\sigma}}{(1+\sigma)I-w^{1+\sigma}}]$ and strictly decreasing on $[\frac{I-w^{1+\sigma}}{(1+\sigma)I-w^{1+\sigma}}, 1]$ and (ii) $\lim_{t\to-\infty} v(t,w) = -\infty$. For any $w \geq ((1+\sigma)I)^{1/(1+\sigma)}$, $v(\cdot, w)$ is strictly decreasing on $(-\infty, 1]$.

[23] Guvenen et al. (2021) have recently provided data on the distribution of lifetime labour incomes. This data is also not ideal for the purposes of the current paper. Remarkably, in the WID data and the Guvenen et al. data, the distribution of income across the population is very similar. The appendix in Ivanov (2022a) elaborates on these points.

[24] $t^*(\cdot)$ is defined in (3). $c_1(\cdot)$ and $c_2(\cdot)$ are defined at the start of section 3.2.

2) $c_1(\cdot)$ *is given by*

$$c_1(t) = \left(\frac{1-(1+\sigma)t}{1-t}I\right)^{1/(1+\sigma)} \tag{10}$$

3) $c_1(\underline{t}) < ((1+\sigma)I)^{1/(1+\sigma)}$.

4) For any $t \in [\underline{t}, \frac{1}{1+\sigma}]$, $v(\cdot, c_1(t))$ *is strictly increasing on* $(-\infty, t]$ *and strictly decreasing on* $[t, 1]$.

5) For any $w \leq c_1(\underline{t})$, $v(\cdot, w)$ *is strictly increasing on* $(-\infty, t^*(w)]$ *and strictly decreasing on* $[t^*(w), 1]$.

6) $c_2(\cdot)$ *is given by*

$$c_2(t) = \begin{cases} c_1(t) & \text{if } t = \underline{t} \\ \left(\frac{t(1-t)^\sigma - \underline{t}(1-\underline{t})^\sigma}{(1-\underline{t})^{1+\sigma} - (1-t)^{1+\sigma}}(1+\sigma)I\right)^{1/(1+\sigma)} & \text{if } \underline{t} < t \leq \frac{1}{1+\sigma} \end{cases}. \tag{11}$$

7) For any $t \in (\underline{t}, \frac{1}{1+\sigma}]$, $v(\underline{t}, w) \leq v(t, w)$ *if and only if* $w \leq c_2(t)$.

8) $c_2(\cdot)$ *is continuous.*

9) $c_2(\cdot)$ *is strictly decreasing.*

10) For any $t \in (\underline{t}, \frac{1}{1+\sigma}]$, $c_2(t) > c_1(t)$.

11) For any $t \in [\underline{t}, \frac{1}{1+\sigma}]$, $c_2(t) \leq c_1(\underline{t})$.

Figure 3 summarises some of the information from Lemma 0.

Proof of Part 1):

Taking the derivative of the right hand-side of (2) yields

$$v_t(t, w) = (1-t)^{\sigma-1}\left(I - w^{1+\sigma} - \left((1+\sigma)I - w^{1+\sigma}\right)t\right). \tag{12}$$

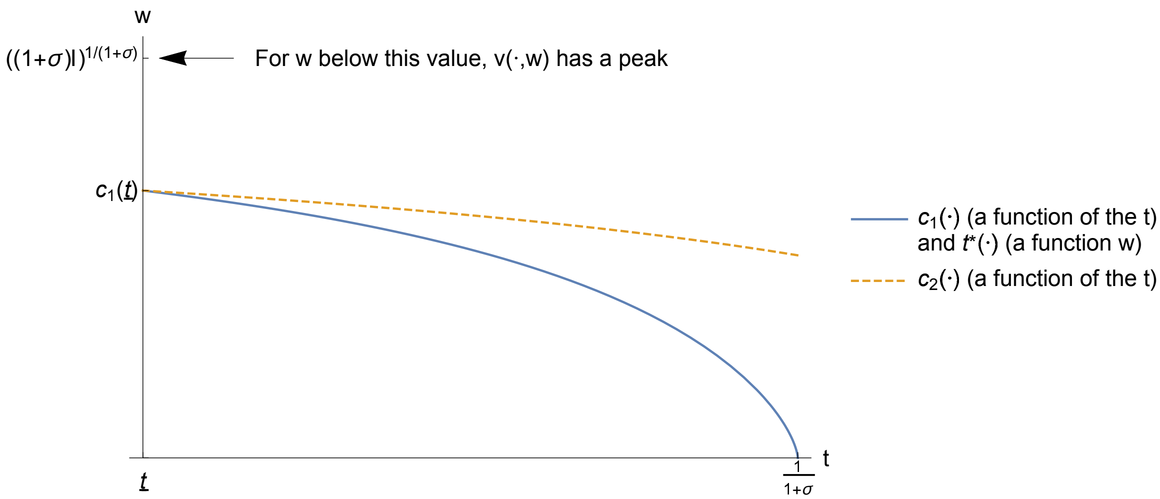

Figure 3: $t^*(\cdot)$ restricted to $w \in [0, \left(\frac{1-(1+\sigma)\underline{t}}{1-\underline{t}} I\right)^{1/(1+\sigma)}]$, $c_1(\cdot)$, and $c_2(\cdot)$.

Note that $v_t(t, w)$ has the same sign as $I - w^{1+\sigma} - ((1+\sigma)I - w^{1+\sigma})\, t$. Thus, if $w < ((1+\sigma)I)^{1/(1+\sigma)}$, $v_t(\cdot, w)$ is strictly positive on $(-\infty, \frac{I-w^{1+\sigma}}{(1+\sigma)I-w^{1+\sigma}})$, zero at $t = \frac{I-w^{1+\sigma}}{(1+\sigma)I-w^{1+\sigma}}$, and strictly negative on $(\frac{I-w^{1+\sigma}}{(1+\sigma)I-w^{1+\sigma}}, 1]$; if $w \geq ((1+\sigma)I)^{1/(1+\sigma)}$, $v_t(\cdot, w)$ is strictly negative on $(-\infty, 1]$.

Also, when $w < ((1+\sigma)I)^{1/(1+\sigma)}$, $\lim_{t\to-\infty} 1 - t = \infty$ and $\lim_{t\to-\infty} (w^{1+\sigma} + ((1+\sigma)I - w^{1+\sigma})\, t) = -\infty$. Thus, one can see from (2) that $\lim_{t\to-\infty} v(t, w) = -\infty$. Q.E.D.

Proof of Part 2):

Expression (10) follows from solving $t = \frac{I-w^{1+\sigma}}{(1+\sigma)I-w^{1+\sigma}}$ for w. Q.E.D.

Proof of Part 3):

Using (10), $c_1(\underline{t}) < ((1+\sigma)I)^{1/(1+\sigma)}$ can be written as

$$\left(\frac{1-(1+\sigma)\underline{t}}{1-\underline{t}} I\right)^{1/(1+\sigma)} < ((1+\sigma)I)^{1/(1+\sigma)},$$

which simplifies to $\sigma > 0$. Q.E.D.

Proof of Part 4):

The fact that $c_1(\cdot)$ is decreasing (see Figure 1) and part 3) imply $c_1(t) < ((1+\sigma)I)^{1/(1+\sigma)}$ for all $t \in [\underline{t}, \frac{1}{1+\sigma}]$. Then, by part 1), $v(\cdot, c_1(t))$ has its peak at $\frac{I - c_1(t)^{1+\sigma}}{(1+\sigma)I - c_1(t)^{1+\sigma}}$. Using (10) to substitute for $c_1(t)$ into the latter expression, we obtain $\frac{I - c_1(t)^{1+\sigma}}{(1+\sigma)I - c_1(t)^{1+\sigma}} = t$. Q.E.D.

Proof of Part 5):

Given (3) and (10), $t^*(w) = \frac{I - w^{1+\sigma}}{(1+\sigma)I - w^{1+\sigma}}$ for $w \leq c_1(\underline{t})$. The statement follows from parts 1) and 3). Q.E.D.

Proof of Part 6):

Expression (11) follows from solving $v(t, w) = v(\underline{t}, w)$ for w when $t \in (\underline{t}, \frac{1}{1+\sigma}]$. Q.E.D.

Proof of Part 7):

Fix $t \in (\underline{t}, \frac{1}{1+\sigma}]$. Using (2), it is straightforward to show that $v(\underline{t}, w) \leq v(t, w)$ can be rewritten as

$$w \leq \left(\frac{t(1-t)^\sigma - \underline{t}(1-\underline{t})^\sigma}{(1-\underline{t})^{1+\sigma} - (1-t)^{1+\sigma}} (1+\sigma) I \right)^{1/(1+\sigma)}.$$

Given (11), the latter inequality is equivalent to $w \leq c_2(t)$. Q.E.D.

Proof of Part 8):

It is clear from (11) that $c_2(\cdot)$ is continuous on $(\underline{t}, \frac{1}{1+\sigma}]$. $c_2(\cdot)$ is also continuous at

$\underline{t}$ because

$$\begin{aligned}
&\lim_{t \downarrow \underline{t}} c_2(t) = \\
&\lim_{t \downarrow \underline{t}} \left(\frac{t(1-t)^\sigma - \underline{t}(1-\underline{t})^\sigma}{(1-\underline{t})^{1+\sigma} - (1-t)^{1+\sigma}} (1+\sigma) I \right)^{1/(1+\sigma)} = \\
&\left((1+\sigma) I \lim_{t \downarrow \underline{t}} \frac{t(1-t)^\sigma - \underline{t}(1-\underline{t})^\sigma}{(1-\underline{t})^{1+\sigma} - (1-t)^{1+\sigma}} \right)^{1/(1+\sigma)} = \\
&\left((1+\sigma) I \lim_{t \downarrow \underline{t}} \frac{(1-t)^{\sigma-1}(1-t-\sigma t)}{(1+\sigma)(1-t)^\sigma} \right)^{1/(1+\sigma)} = \\
&\left(I \lim_{t \downarrow \underline{t}} \frac{(1-(1+\sigma)t)}{(1-t)} \right)^{1/(1+\sigma)} = \\
&\left(I \frac{(1-(1+\sigma)\underline{t})}{(1-\underline{t})} \right)^{1/(1+\sigma)} = c_1(\underline{t}) = c_2(\underline{t}),
\end{aligned}$$

where the third equality uses l'Hópital's rule. Q.E.D.

Proof of Part 9):

Take $t', t \in (\underline{t}, \frac{1}{1+\sigma}]$ such that $t' > t$. We have $v(\underline{t}, c_2(t)) = v(t, c_2(t)) > v(t', c_2(t))$, where the equality follows from the definition of $c_2(t)$ and the inequality follows because, given part 1), $v(\underline{t}, c_2(t)) = v(t, c_2(t))$ implies that the peak of $v(\cdot, c_2(t))$ must occur between $\underline{t}$ and t. From part 7), $v(\underline{t}, c_2(t)) > v(t', c_2(t))$ if and only if $c_2(t) > c_2(t')$. Thus, $c_2(\cdot)$ is strictly decreasing on $(\underline{t}, \frac{1}{1+\sigma}]$. Moreover, because $c_2(\cdot)$ is continuous at $\underline{t}$ (by part 8)), $c_2(\cdot)$ is strictly decreasing on $[\underline{t}, \frac{1}{1+\sigma}]$. Q.E.D.

Proof of Part 10):

Fix $t \in (\underline{t}, \frac{1}{1+\sigma}]$. Given part 1), $v(\underline{t}, c_2(t)) = v(t, c_2(t))$ implies that the peak of $v(\cdot, c_2(t))$ must occur strictly between $\underline{t}$ and t. Given part 4), the peak of $v(\cdot, c_1(t))$ occurs at t. Thus, given part 1), we must have $\frac{I - c_2(t)^{1+\sigma}}{(1+\sigma)I - c_2(t)^{1+\sigma}} < \frac{I - c_1(t)^{1+\sigma}}{(1+\sigma)I - c_1(t)^{1+\sigma}}$. Given that (again by part 1)) $(1+\sigma)I > c_2(t)^{1+\sigma}$ and $(1+\sigma)I > c_1(t)^{1+\sigma}$, the inequality in the last sentence can be rewritten as $c_2(t) > c_1(t)$. Q.E.D.

Proof of Part 11):

We have $c_2(t) \leq c_2(\underline{t}) = c_1(\underline{t})$, where the inequality follows from part 9). Q.E.D.

8.2 Proof of Lemma 1

Proof of part 1):

Given any $t \in [\underline{t}, \overline{t})$, we have

$$v(t,w) = \frac{1}{1+\sigma}(1-t)^{1+\sigma}w^{1+\sigma} + t(1-t)^{\sigma}I - R \geq$$
$$\frac{1}{1+\sigma}(1-t)^{1+\sigma}w^{1+\sigma} > \frac{1}{1+\sigma}(1-\overline{t})^{1+\sigma}w^{1+\sigma} =$$
$$\frac{1}{1+\sigma}(1-\overline{t})^{1+\sigma}w^{1+\sigma} + \overline{t}(1-\overline{t})^{\sigma}I - R = v(\overline{t}, w).$$

The first and last equality use (1), plugging in for $l(w)$. The first inequality uses the fact that the UBI is nonnegative for any tax rate $t \in [\underline{t}, \overline{t})$. The penultimate equality uses the fact that the UBI for tax rate $\overline{t}$ equals zero. Q.E.D.

Proof of part 2):

Given part 1) of Lemma 0, type w's optimal feasible tax rate equals $\frac{I-w^{1+\sigma}}{(1+\sigma)I-w^{1+\sigma}}$ if $w < ((1+\sigma)I)^{1/(1+\sigma)}$ and $w \leq \left(\frac{1-(1+\sigma)\underline{t}}{1-\underline{t}}I\right)^{1/(1+\sigma)}$ and equals $\underline{t}$ otherwise.[25,26]

Given $c_1(\underline{t}) = \left(\frac{1-(1+\sigma)\underline{t}}{1-\underline{t}}I\right)^{1/(1+\sigma)}$ (see (10)) and part 3) of Lemma 0, $w \leq \left(\frac{1-(1+\sigma)\underline{t}}{1-\underline{t}}I\right)^{1/(1+\sigma)}$ implies $w < ((1+\sigma)I)^{1/(1+\sigma)}$. Thus, type w's optimal feasible tax rate is given by $t^*(\cdot)$ as defined in (3). Q.E.D.

Proof of part 3):

[25] $w \leq \left(\frac{1-(1+\sigma)\underline{t}}{1-\underline{t}}I\right)^{1/(1+\sigma)}$ is obtained by rearranging $\frac{I-w^{1+\sigma}}{(1+\sigma)I-w^{1+\sigma}} \geq \underline{t}$.

[26] Note that $\frac{I-w^{1+\sigma}}{(1+\sigma)I-w^{1+\sigma}} \leq \frac{1}{1+\sigma} < \overline{t}$, where the first inequality holds because $\frac{I-w^{1+\sigma}}{(1+\sigma)I-w^{1+\sigma}}$ is maximised at $w=0$ (the derivative of $\frac{I-w^{1+\sigma}}{(1+\sigma)I-w^{1+\sigma}}$ with respect to w equals $\frac{-\sigma(1+\sigma)Iw^{\sigma}}{((1+\sigma)I-w^{1+\sigma})^2} < 0$) and the second inequality holds because $\overline{t}$ is above the revenue-maximising rate, $\frac{1}{1+\sigma}$.

The only possible discontinuity of $t^*(\cdot)$ would be at $w = \left(\frac{1-(1+\sigma)\underline{t}}{1-\underline{t}} I\right)^{1/(1+\sigma)}$. However, by plugging into (3), one can directly verify that $t^*\left(\left(\frac{1-(1+\sigma)\underline{t}}{1-\underline{t}} I\right)^{1/(1+\sigma)}\right) = \underline{t}$.

That $t^*(\cdot)$ is strictly decreasing on $[0, \left(\frac{1-(1+\sigma)\underline{t}}{1-\underline{t}} I\right)^{1/(1+\sigma)}]$ follows from the fact that the derivative of $\frac{I-w^{1+\sigma}}{(1+\sigma)I-w^{1+\sigma}}$ with respect to w equals $\frac{-\sigma(1+\sigma)Iw^{\sigma}}{((1+\sigma)I-w^{1+\sigma})^2} < 0$. Q.E.D.

Proof of part 4):

From the proof of part 2), it is clear that $t^*(w)$ can be written as

$$t^*(w) = \begin{cases} \frac{I-w^{1+\sigma}}{(1+\sigma)I-w^{1+\sigma}} & \text{if } w < ((1+\sigma)I)^{1/(1+\sigma)}, \frac{I-w^{1+\sigma}}{(1+\sigma)I-w^{1+\sigma}} \geq \underline{t} \\ \underline{t} & \text{otherwise} \end{cases}.$$

Part 4) then follows from part 1) of Lemma 0. Q.E.D.

8.3 Proof of Lemma 2

Proof of part 1):

This has already been proved–see part 7) of Lemma 0. Q.E.D.

Proof of part 2):

This has already been proved–see part 10) of Lemma 0. Q.E.D.

Proof of part 3):

Given part 11) of Lemma 0, $F(c_2(t)) \leq F(c_1(\underline{t}))$ must hold for any $t \in [\underline{t}, \frac{1}{1+\sigma}]$. To complete the proof, I now show that $F(c_1(\underline{t})) < 1$.

Suppose $F(c_1(\underline{t})) = 1$. Then, for any type w in the support of F, we have

$$w^{1+\sigma} \leq \frac{1-(1+\sigma)\underline{t}}{1-\underline{t}} I.$$

Summing over the population, yields

$$\int_0^\infty w^{1+\sigma} dF(w) \leq \frac{1-(1+\sigma)\underline{t}}{1-\underline{t}} I,$$

which, given the definition of I is equivalent to

$$1 \leq \frac{1-(1+\sigma)\underline{t}}{1-\underline{t}}.$$

The latter inequality can be rewritten as $\sigma\underline{t} \leq 0$, a contradiction. Q.E.D.

Proof of part 4):

Fix $t \in [\underline{t}, \frac{1}{1+\sigma}]$ and $w \in [0, c_1(t)) \cup (c_1(t), c_2(t)]$ such that $w \neq c_1(t)$ (or, equivalently, such that $t \neq t^*(w)$–see Figure 3). We have $(1+\sigma)I > w^{1+\sigma}$ (again, see Figure 3) so that, by parts 1) and 5) of Lemma 0, $v(\cdot, w)$ has a peak at $t^*(w) = \frac{I-w^{1+\sigma}}{(1+\sigma)I-w^{1+\sigma}}$. From (12), it is clear that $v_t(t', w) \neq 0$ for any $t' \neq t^*(w)$. Moreover, given that the peak of $v(\cdot, w)$ occurs at $t^*(w)$, it must be that t and $\tau(t, w)$ lie on opposite sides of $t^*(w)$, so that $\tau(t, w) \neq t^*(w)$, $v_t(\tau(t, w), w) \neq 0$, and $\frac{v_t(t,w)}{v_t(\tau(t,w),w)} < 0$. Q.E.D.

8.4 Proof of Lemma 3

I will consider separately the cases $t > \underline{t}$ and $t = \underline{t}$. The proof is based on Figure 2 (which pertains to the case $t > \underline{t}$) and Figure 4 (which pertains to the case $t = \underline{t}$). Note that part 4) of Lemma 1 ensures the single-peakedness of the v functions in these figures.

Case $t > \underline{t}$:

First, consider type $w_1 \leq c_1(t)$. Then, $v(\cdot, w_1)$ is as in Figure 2. In particular, based on Figure 1, $t^*(w_1) \geq t$ (with strict inequality if $w_1 < c_1(t)$) and, by part 1) in Lemma 1, $v(t, w_1) > v(\overline{t}, w_1)$. From Figure 2, it is obvious that $\alpha(t, w_1) = \tau(t, w_1) - t$.

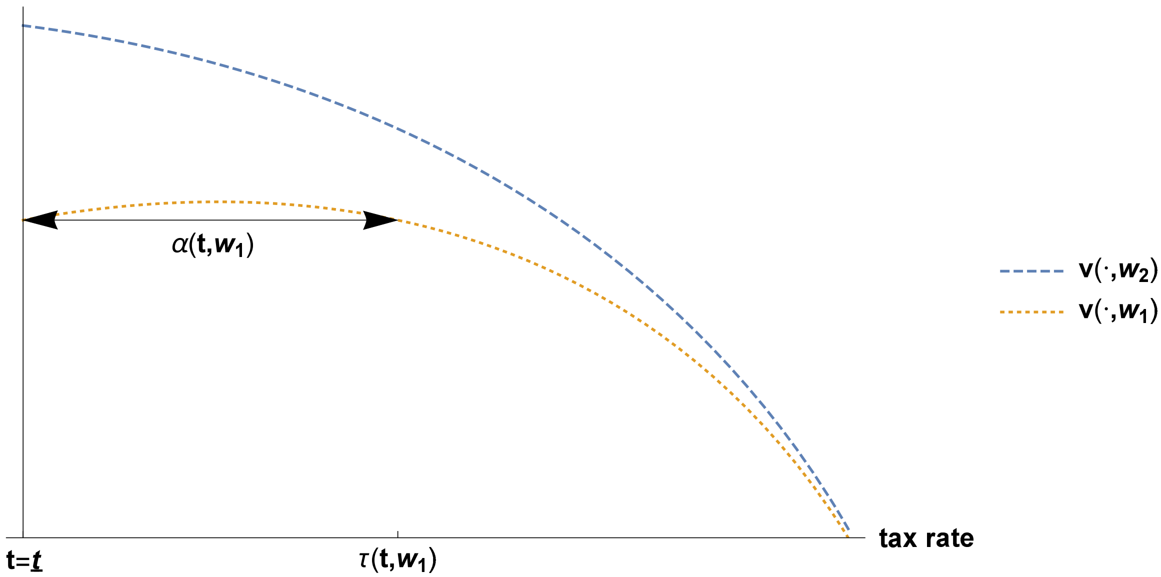

Figure 4: $\alpha(t, w_1)$ and $\alpha(t, w_2)$, where $t = \underline{t}$ and $w_1 < c_1(t) \leq w_2$.

Next, consider type w_2 such that $c_1(t) < w_2 \leq c_2(t)$. Then, $v(\cdot, w_2)$ is as in Figure 2. In particular, $c_1(t) < w_2$ ensures that $t^*(w_2) < t$ (see Figure 1) and part 1) of Lemma 2 guarantees that $v(\underline{t}, w_2) \leq v(t, w_2)$. From Figure 2, it is obvious that $\alpha(t, w_2) = t - \tau(t, w_2)$.

Finally, consider type $w_3 > c_2(t)$. Then, $v(\cdot, w_3)$ is as in Figure 2. In particular, by part 2) of Lemma 2, $w_3 > c_1(t)$, which implies $t^*(w_3) < t$ (see Figure 1). By part 1) of Lemma 2, $v(\underline{t}, w_3) > v(t, w_3)$. From Figure 2, it is obvious that $\alpha(t, w_3) = t - \underline{t}$.

Case $t = \underline{t}$*:*

In this case, expression (5) simplifies to

$$\alpha(t, w) = \begin{cases} \tau(t, w) - t & \text{if } w \leq c_1(t) \\ 0 & \text{if } w > c_1(t) \end{cases}. \tag{13}$$

By part 4) of Lemma 0, $v(\cdot, c_1(t))$ has its a peak at t so that $\tau(t, c_1(t)) = t$. Thus,

(13) can be rewritten as

$$\alpha(t,w)=\begin{cases}\tau(t,w)-t & \text{if } w<c_1(t)\\ 0 & \text{if } w\geq c_1(t)\end{cases}. \tag{14}$$

First, consider type $w_1 < c_1(t)$. Then, $v(\cdot, w_1)$ is as in Figure 4. In particular, based on Figure 1, $t^*(w_1) > t$ and, by part 1) in Lemma 1, $v(t, w_1) > v(\overline{t}, w_1)$. From Figure 4, it is obvious that $\alpha(t, w_1) = \tau(t, w_1) - t$.

Next, consider type $w_2 \geq c_1(t)$. Then, $v(\cdot, w_2)$ is as in Figure 4. In particular, $t^*(w_2) = \underline{t}$ (see Figure 1). From Figure 4, it is obvious that $\alpha(t, w_2) = 0$. Q.E.D.

8.5 Proof of Lemma 4

I first state and prove the following claim.

Claim 1 *τ is continuous at any point in* $\{(t,w)|\underline{t} \leq t \leq \frac{1}{1+\sigma}, 0 \leq w \leq c_2(t)\}$.

Proof:

Consider a point (t, w), where $\underline{t} \leq t \leq \frac{1}{1+\sigma}$ and $0 \leq w \leq c_2(t)$, and let $\{(t_n, w_n)\}_{n=1}^{\infty}$ be such that $(t_n, w_n) \to (t, w)$.

Note that $\{\tau(t_n, w_n)\}_{n=1}^{\infty}$ is bounded based on the following argument. Take $\hat{w}$ such that $c_2(t) < \hat{w} < ((1+\sigma)I)^{1/(1+\sigma)}$.[27] Given that v is strictly increasing in its second argument (see (2)), we have $v(\cdot, \hat{w}) > v(\cdot, w_n)$ for all large enough n. Also, $v(t_n, w_n) > 0$ for all large enough n because v is continuous and $v(t, w) > v(\overline{t}, w) = \frac{1}{1+\sigma}(1-\overline{t})^{1+\sigma}w^{1+\sigma} > 0$ (that the equality holds can be seen in the proof of part 1) of Lemma 1). Thus, for all large enough n, $\tau(t_n, w_n)$ is greater than the smaller of the two values of $\tilde{t}$ that solve $v(\tilde{t}, \hat{w}) = 0$.[28] Thus, $\{\tau(t_n, w_n)\}_{n=1}^{\infty}$ is bounded from below.

[27] We have $c_2(t) \leq c_1(\underline{t}) < ((1+\sigma)I)^{1/(1+\sigma)}$, where the first and last inequalities follow from parts 11) and 3), respectively, of Lemma 0. Thus, $\hat{w}$ can be picked in this way.

[28] $\hat{w} < ((1+\sigma)I)^{1/(1+\sigma)}$ guarantees that, apart from $\tilde{t} = 1$, there is another, smaller value of $\tilde{t}$ that solves $v(\tilde{t}, \hat{w}) = 0$ (see part 1) of Lemma 0).

It is also clearly bounded from above given that $\tau(t_n, w_n) \leq 1$ for all n.

Also, note that, by the continuity of v, $v(t_n, w_n) \to v(t, w)$. Hence, given that $v(t_n, w_n) = v(\tau(t_n, w_n), w_n)$, we must have $v(\tau(t_n, w_n), w_n) \to v(t, w)$.

Now, suppose $\tau(t_n, w_n)$ doesn't converge to $\tau(t, w)$. Then, given that $\{\tau(t_n, w_n)\}_{n=1}^{\infty}$ is bounded, it has a subsequence, $\{\tau(t_{n_k}, w_{n_k})\}_{k=1}^{\infty}$, that converges to $z \neq \tau(t, w)$, so that, by the continuity of v, $v(\tau(t_{n_k}, w_{n_k}), w_{n_k}) \to v(z, w)$ as $k \to \infty$.

Consider the following four exhaustive cases. First, if $z \neq t$, then, given that also $z \neq \tau(t, w)$, we must have $v(z, w) \neq v(t, w)$, which contradicts $v(\tau(t_n, w_n), w_n) \to v(t, w)$. Second, if $z = t = t^*(w)$, then, given part 5) of lemma 0, $\tau(t, w) = z$, a contradiction. Third, if $z = t > t^*(w)$, then, given the continuity of $t^*(\cdot)$, $t^*(w_{n_k}) < t_{n_k}$ for all large enough k. Hence, $\tau(t_{n_k}, w_{n_k}) < t^*(w_{n_k}) < t_{n_k}$ for all large enough k.[29] Taking limits as $k \to \infty$ and using the continuity of $t^*(\cdot)$, we get that $z \leq t^*(w) \leq t$, which contradicts $z = t > t^*(w)$. Fourth, if $z = t < t^*(w)$, then, given the continuity of $t^*(\cdot)$, $t_{n_k} < t^*(w_{n_k})$ and $t < t^*(w_{n_k})$ for all large enough k. The latter inequality implies $\underline{t} < t^*(w_{n_k})$ so that $w_{n_k} < c_1(\underline{t})$ (see Figure 1) and, hence, by part 5) of Lemma 0, the peak of $v(\cdot, w_{n_k})$ lies at $t^*(w_{n_k})$. Hence, $t_{n_k} < t^*(w_{n_k}) < \tau(t_{n_k}, w_{n_k})$ for all large enough k.[30] Taking limits as $k \to \infty$ and using the continuity of $t^*(\cdot)$, we get that $t \leq t^*(w) \leq z$, which contradicts $z = t < t^*(w)$.

Thus, we must have $\tau(t_n, w_n) \to \tau(t, w)$. Q.E.D.

Under Assumption 1, $B(t, F)$ in (6) can be written as

$$B(t, F) = -\int_0^{c_1(t)} (\tau(t, w) - t) f(w) dw - \int_{c_1(t)}^{c_2(t)} (t - \tau(t, w)) f(w) dw - (t - \underline{t}) \int_{c_2(t)}^{\infty} f(w) dw$$

[29]For all large enough k, w_{n_k} will be close enough to w that $w_{n_k} < ((1+\sigma)I)^{1/(1+\sigma)}$ holds (recall that $c_2(t) \leq c_1(\underline{t}) < ((1+\sigma)I)^{1/(1+\sigma)}$). $w_{n_k} < ((1+\sigma)I)^{1/(1+\sigma)}$ guarantees that $v(\cdot, w_{n_k})$ has an inverse-U shape with a peak that lies weakly to the left of $t^*(w_{n_k})$ (see part 1) of Lemma 0). Thus, $\tau(t_{n_k}, w_{n_k})$ and t_{n_k} will lie on opposite sides of $t^*(w_{n_k})$.

[30]For all large enough k, t_{n_k} will be close enough to t that $v(t_{n_k}, w_{n_k}) > v(\bar{t}, w_{n_k})$. Thus, $\tau(t_{n_k}, w_{n_k})$ and t_{n_k} will lie on opposite sides of $t^*(w_{n_k})$.

Given Claim 1, the fact that $c_1(\cdot)$ is continuous (see 10), and the fact that $c_2(\cdot)$ is continuous (by part 8) of Lemma 0), the first two integrals above are continuous on $[\underline{t}, \frac{1}{1+\sigma}]$ (see Theorem 28 and the discussion on p.369 in Erwe (1967)). Hence, $B(\cdot, F)$ is continuous on $[\underline{t}, \frac{1}{1+\sigma}]$. Hence, $B(\cdot, F)$ has a maximum on the closed and bounded interval $[\underline{t}, \frac{1}{1+\sigma}]$. Q.E.D.

8.6 Proof of Proposition 1

The proof is based on a sequence of claims.

Claim 1 *$c_1(\cdot)$ and $c_2(\cdot)$ are continuously differentiable on $[\underline{t}, \frac{1}{1+\sigma}]$.*[31]

Proof:

From the functional forms of $c_1(\cdot)$ and $c_2(\cdot)$ in (10) and (11), it is obvious that $c_1(\cdot)$ and $c_2(\cdot)$ are continuously differentiable on $[\underline{t}, \frac{1}{1+\sigma}]$ and $(\underline{t}, \frac{1}{1+\sigma}]$, respectively.

It remains to show that $c_2'(\cdot)$ exists and is continuous at $\underline{t}$. We have

$$\begin{aligned}
&\lim_{t \downarrow \underline{t}} \frac{c_2(t) - c_2(\underline{t})}{t - \underline{t}} = \lim_{t \downarrow \underline{t}} c_2'(t) = \\
&- \lim_{t \downarrow \underline{t}} \frac{I(1-t)^{\sigma-1}\left((1-t)^{1+\sigma} + (1-\underline{t})^{\sigma}(\sigma t + t - \sigma \underline{t} - 1)\right)}{\left(\frac{t(1-t)^{\sigma} - \underline{t}(1-\underline{t})^{\sigma}}{(1-\underline{t})^{1+\sigma} - (1-t)^{1+\sigma}}(1+\sigma)I\right)^{\frac{\sigma}{1+\sigma}} \left((1-t)^{1+\sigma} - (1-\underline{t})^{1+\sigma}\right)^2} = \\
&- I(1-\underline{t})^{\sigma-1} \lim_{t \downarrow \underline{t}} \frac{1}{c_2(t)^{\sigma}} \lim_{t \downarrow \underline{t}} \frac{(1-t)^{1+\sigma} + (1-\underline{t})^{\sigma}(\sigma t + t - \sigma \underline{t} - 1)}{\left((1-t)^{1+\sigma} - (1-\underline{t})^{1+\sigma}\right)^2} = \\
&- \frac{I(1-\underline{t})^{\sigma-1}}{c_2(\underline{t})^{\sigma}} \lim_{t \downarrow \underline{t}} \frac{(1-t)^{1+\sigma} + (1-\underline{t})^{\sigma}(\sigma t + t - \sigma \underline{t} - 1)}{\left((1-t)^{1+\sigma} - (1-\underline{t})^{1+\sigma}\right)^2} = \\
&- \frac{I}{2c_2(\underline{t})^{\sigma}(1-\underline{t})} \lim_{t \downarrow \underline{t}} \frac{(1-t)^{\sigma} - (1-\underline{t})^{\sigma}}{(1-t)^{1+\sigma} - (1-\underline{t})^{1+\sigma}} = \\
&- \frac{I}{2c_2(\underline{t})^{\sigma}(1-\underline{t})} \lim_{t \downarrow \underline{t}} \frac{\sigma}{(1+\sigma)(1-t)} = -\frac{\sigma I}{2(1+\sigma)c_2(\underline{t})^{\sigma}(1-\underline{t})^2},
\end{aligned}$$

[31] $c_1'(\underline{t})$ and $c_2'(\underline{t})$ are to be understood as right derivatives. $c_1'(\frac{1}{1+\sigma})$ and $c_2'(\frac{1}{1+\sigma})$ are to be understood as left derivatives.

where the first, fifth, and sixth equalities use l'Hópital's rule.[32] Thus, $c_2(\cdot)$ is differentiable at $\underline{t}$. Moreover, because $c_2'(\underline{t}) = \lim_{t \downarrow \underline{t}} c_2'(t)$ (as is evident from the first equality above), $c_2'(\cdot)$ is continuous at $\underline{t}$. Q.E.D.

Claim 2 *At any point in* $\{(t,w)|\underline{t} \leq t \leq \frac{1}{1+\sigma}, 0 \leq w \leq c_2(t)\}$*,* τ_t *exists and is continuous.*

Proof:

By the implicit function theorem, τ_t exists and is continuous on $\{(t,w)|\underline{t} \leq t \leq \frac{1}{1+\sigma}, 0 \leq w \leq c_2(t), w \neq c_1(t)\}$ because, as can be seen from the proof of part 4) of Lemma 2, in this region $v_t(\tau(t,w),w) \neq 0$.

It remains to show that $\tau_t(t, c_1(t))$ exists for any $t \in [\underline{t}, \frac{1}{1+\sigma}]$ or, equivalently (see Figure 3), that $\tau_t(t^*(w), w)$ exists for any $w \in [0, c_1(\underline{t})]$. Assumption 2 will then imply that, for any $w \in [0, c_1(\underline{t})]$, τ_t is continuous at $(t^*(w), w)$.

I shall first show that, for any $w \in [0, c_1(\underline{t})]$, $\lim_{t \uparrow t^*(w)} \tau_t(t,w)$ and $\lim_{t \downarrow t^*(w)} \tau_t(t,w)$ exist and are both equal to -1. Starting with the former limit, we have:

$$\lim_{t \uparrow t^*(w)} \tau_t(t,w) = \lim_{t \uparrow t^*(w)} \frac{v_t(t,w)}{v_t(\tau(t,w),w)} =$$
$$\lim_{t \uparrow t^*(w)} \frac{(1-t)^{\sigma-1}\left(I - w^{1+\sigma} - ((1+\sigma)I - w^{1+\sigma})t\right)}{(1-\tau(t,w))^{\sigma-1}\left(I - w^{1+\sigma} - ((1+\sigma)I - w^{1+\sigma})\tau(t,w)\right)} =$$
$$\lim_{t \uparrow t^*(w)} \frac{(1-t)^{\sigma-1}}{(1-\tau(t,w))^{\sigma-1}} \lim_{t \uparrow t^*(w)} \frac{I - w^{1+\sigma} - ((1+\sigma)I - w^{1+\sigma})t}{I - w^{1+\sigma} - ((1+\sigma)I - w^{1+\sigma})\tau(t,w)} =$$
$$\lim_{t \uparrow t^*(w)} \frac{I - w^{1+\sigma} - ((1+\sigma)I - w^{1+\sigma})t}{I - w^{1+\sigma} - ((1+\sigma)I - w^{1+\sigma})\tau(t,w)} =$$
$$\lim_{s \downarrow 1} \frac{I - w^{1+\sigma} - ((1+\sigma)I - w^{1+\sigma})\hat{t}(s)}{I - w^{1+\sigma} - ((1+\sigma)I - w^{1+\sigma})\tau(\hat{t}(s),w)}, \tag{15}$$

[32] The continuity of $c_2(\cdot)$ at $\underline{t}$ (see part 8) of Lemma 0) implies that $\lim_{t \downarrow \underline{t}} c_2(t) - c_2(\underline{t}) = 0$ so that l'Hópital's rule can be used in the first equality.

where the first equality above uses the implicit function theorem and

$$\hat{t}(s) = \frac{(1+\sigma)I\frac{s^{1/\sigma}-1}{s^{1/\sigma+1}-1} - w^{1+\sigma}}{(1+\sigma)I - w^{1+\sigma}}.$$

The equality between the last two limits above holds because (i) $\hat{t}(\cdot)$ is strictly decreasing for $s > 1$ and (ii) $\lim_{s\downarrow 1} \hat{t}(s) = \frac{I-w^{1+\sigma}}{(1+\sigma)I-w^{1+\sigma}} = t^*(w)$ (because, by l'Hópital's rule, $\lim_{s\downarrow 1} \frac{s^{1/\sigma}-1}{s^{1/\sigma+1}-1} = 1/(1+\sigma)$).

Furthermore, letting

$$\hat{\tau}(s) = \frac{(1+\sigma)I\frac{s^{1/\sigma}-1}{s^{1/\sigma+1}-1}s - w^{1+\sigma}}{(1+\sigma)I - w^{1+\sigma}},$$

it is straightforward to verify that, for $s > 1$, $\hat{\tau}(s) \neq \hat{t}(s)$ and $v(\hat{\tau}(s), w) = v(\hat{t}(s), w)$. Thus, $\tau(\hat{t}(s), w) = \hat{\tau}(s)$.[33] Thus, (15) can be written as:

$$\begin{aligned}
&\lim_{s\downarrow 1} \frac{I - w^{1+\sigma} - ((1+\sigma)I - w^{1+\sigma})\hat{t}(s)}{I - w^{1+\sigma} - ((1+\sigma)I - w^{1+\sigma})\hat{\tau}(s)} = \\
&\lim_{s\downarrow 1} \frac{1 - (1+\sigma)\frac{s^{1/\sigma}-1}{s^{1/\sigma+1}-1}}{1 - (1+\sigma)\frac{s^{1/\sigma}-1}{s^{1/\sigma+1}-1}s} = \\
&\lim_{s\downarrow 1} \frac{(1+\sigma)s^{1/\sigma} - s^{1/\sigma+1} - \sigma}{1 - (1+\sigma)s + \sigma s^{1/\sigma+1}} = \\
&\lim_{s\downarrow 1} \frac{s^{1/\sigma-1}(1-s)}{\sigma(s^{1/\sigma}-1)} = \\
&\lim_{s\downarrow 1} \frac{s^{1/\sigma-1}}{\sigma} \lim_{s\downarrow 1} \frac{1-s}{s^{1/\sigma}-1} = \\
&\frac{1}{\sigma} \lim_{s\downarrow 1} \frac{1-s}{s^{1/\sigma}-1} = \\
&\frac{1}{\sigma} \lim_{s\downarrow 1} -\sigma s^{1-1/\sigma} = -1
\end{aligned}$$

[33] $(\hat{t}(s), \hat{\tau}(s))$ parameterises the graph of $\tau(\cdot, w)$ around $t = t^*(w)$ (excluding $t = t^*(w)$). I am grateful to an anonymous user (with username Maxim) on the StackExchange Mathematics online forum for suggesting this technique to me. See URL (version: 2022-03-17): https://math.stackexchange.com/q/4405430

The third and sixth equalities above make use of l'Hópital's rule.

An analogous calculation (that uses the same $\hat{t}(\cdot)$ and $\hat{\tau}(\cdot)$) establishes that $\lim_{t\downarrow t^*(w)} \tau_t(t,w) = -1$.

Now, given $w \in [0, c_1(\underline{t})]$, (i) the left derivative of $\tau(\cdot, w)$ at $t^*(w)$ equals

$$\lim_{t\uparrow t^*(w)} \frac{\tau(t,w) - \tau(t^*(w),w)}{t - t^*(w)} = \lim_{t\uparrow t^*(w)} \tau_t(t,w) = -1,$$

where the first equality follows from l'Hópital's rule, and (ii) the right derivative of $\tau(\cdot, w)$ at $t^*(w)$ equals

$$\lim_{t\downarrow t^*(w)} \frac{\tau(t,w) - \tau(t^*(w),w)}{t - t^*(w)} = \lim_{t\downarrow t^*(w)} \tau_t(t,w) = -1,$$

where the first equality follows from l'Hópital's rule.[34] Thus, the left and right derivatives of $\tau(\cdot, w)$ at $t^*(w)$ exist and are equal. Hence, $\tau_t(t^*(w), w)$ exists.[35] Q.E.D.

Claim 3 *For any* $t \in [\underline{t}, \frac{1}{1+\sigma}]$,[36]

$$B_t(t,F) = \int_0^{c_1(t)} \left(1 - \frac{v_t(t,w)}{v_t(\tau(t,w),w)}\right) f(w)dw - \int_{c_1(t)}^{c_2(t)} \left(1 - \frac{v_t(t,w)}{v_t(\tau(t,w),w)}\right) f(w)dw - \int_{c_2(t)}^{\infty} f(w)dw. \tag{16}$$

Proof:

[34] The continuity of τ implies that $\lim_{t\uparrow t^*(w)} \tau(t,w) - \tau(t^*(w),w) = \lim_{t\downarrow t^*(w)} \tau(t,w) - \tau(t^*(w),w) = 0$, so that l'Hópital's rule can be used in (i) and (ii).

[35] Note that, because $\lim_{t\uparrow t^*(w)} \tau_t(t,w) = \lim_{t\downarrow t^*(w)} \tau_t(t,w) = \tau_t(t^*(w),w)$, τ_t is continuous at $(t^*(w), w)$ in the direction of the first argument. Also, because $\tau_t(t^*(w),w) = -1$ for all $w \in [0, c_1(\underline{t})]$, τ_t is continuous along the $c_1(\cdot)$ curve in t-w space (see Figure 3).

[36] $B_t(\underline{t}, F)$ and $B_t(\frac{1}{1+\sigma}, F)$ are to be interpreted as a right and left derivative, respectively.

Expression (6) can be rewritten as follows in terms of f rather than F.

$$B(t,F)=\int_0^{c_1(t)}(t-\tau(t,w))f(w)dw-\int_{c_1(t)}^{c_2(t)}(t-\tau(t,w))f(w)dw-(t-\underline{t})\int_{c_2(t)}^{\infty}f(w)dw \tag{17}$$

Claim 1 in the proof of Lemma 4 and Claims 1 and 2 above ensure that, on $[\underline{t},\frac{1}{1+\sigma}]$, $B(\cdot,F)$ is differentiable and the Leibniz integral rule can be used to obtain the derivative of (17).[37] Straightforward application of the Leibniz integral rule to (17) then yields:

$$B_t(t,F)=(t-\tau(t,c_1(t)))f(c_1(t))c_1'(t)+\int_0^{c_1(t)}(1-\tau_t(t,w))f(w)dw-$$
$$(t-\tau(t,c_2(t)))f(c_2(t))c_2'(t)+(t-\tau(t,c_1(t)))f(c_1(t))c_1'(t)-\int_{c_1(t)}^{c_2(t)}(1-\tau_t(t,w))f(w)dw-$$
$$\int_{c_2(t)}^{\infty}f(w)dw+(t-\underline{t})f(c_2(t))c_2'(t)=$$
$$\int_0^{c_1(t)}(1-\tau_t(t,w))f(w)dw-\int_{c_1(t)}^{c_2(t)}(1-\tau_t(t,w))f(w)dw-\int_{c_2(t)}^{\infty}f(w)dw,$$

where the second equality holds because $\tau(t,c_2(t))=\underline{t}$ and, by part 4) of Lemma 0, $\tau(t,c_1(t))=t$.

Finally, by the implicit function theorem, $\tau_t(t,w)=\frac{v_t(t,w)}{v_t(\tau(t,w),w)}$ on $\{(t,w)|\underline{t}\le t\le \frac{1}{1+\sigma},0\le w\le c_2(t),w\ne c_1(t)\}$ because in this region, as can be seen from the proof of part 4) of Lemma 2, $v_t(\tau(t,w),w)\ne 0$. Q.E.D.

Given Claim 3, (8) is a standard first-order condition. Note that, at $t=\frac{1}{1+\sigma}$, $c_1(t)=0$ (see (10)) and $\frac{v_t(t,w)}{v_t(\tau(t,w),w)}<0$ for all $0<w\le c_2(t)$ (recall part 4) of Lemma 2) so that $B_t(t,F)=-\int_0^{c_2(t)}\left(1-\frac{v_t(t,w)}{v_t(\tau(t,w),w)}\right)f(w)dw-\int_{c_2(t)}^{\infty}f(w)dw<0$. Thus, we need not be concerned that $t=\frac{1}{1+\sigma}$ is BO and $B_t(t,F)>0$ at that corner. Q.E.D.

[37] See page 476 in Wrede and Spiegel (2010).

8.7 Proof of Lemma 5

Suppose that $t \in [\underline{t}, \overline{t}]$ is such that $t > t^*(w_{50})$. Then, any type $w \leq w_{50}$ strictly prefers $t^*(w_{50})$ to t given that $t^*(w) \leq t^*(w_{50})$ (see part 3) in Lemma 1) and $v(\cdot, w)$ is single-peaked on $[\underline{t}, \overline{t}]$ (see part 4) in Lemma 1). Furthermore, given the continuity of $v(\cdot, \cdot)$ in its second argument, one can find $\hat{w} > w_{50}$ such that, for all $w \in (w_{50}, \hat{w})$, $v(t^*(w_{50}), w) > v(t, w)$. Thus, strictly more than half the population strictly prefers $t^*(w_{50})$ to t.[38]

The case with $t < t^*(w_{50})$ instead of $t > t^*(w_{50})$ is analogous. Q.E.D.

8.8 Proof of Proposition 3

To complete the proof given in the main text, it remains to show that (i) Assumption 2 holds and (ii) the $\frac{v_t(t,w)}{v_t(\tau(t,w),w)}$ terms under the integrals in (7) equal -1.

Suppose $w \leq c_1(\underline{t})$. Then, one can obtain directly from the definition of $\tau(t, w)$ that $\tau(t, w) = \frac{2I-2w^2}{2I-w^2} - t$ so that $\tau_t(t, w) = -1$.[39] Thus, Assumption 2 holds.

Next, take $t \in [\underline{t}, \frac{1}{1+\sigma}]$ and $w \in [0, c_1(t)) \cup (c_1(t), c_2(t)]$. Again, $\tau(t, w) = \frac{2I-2w^2}{2I-w^2} - t$.[40] Directly plugging into (12), yields $\frac{v_t(t,w)}{v_t(\tau(t,w),w)} = -\frac{I-w^2-(2I-w^2)t}{I-w^2-(2I-w^2)t} = -1$.[41] Q.E.D.

[38] The assumption that there is a unique median type implies that $F(\cdot)$ is strictly increasing around that type so that there is a positive mass of types in $(w_{50}, \hat{w})$.

[39] Part 3) of Lemma 0 guarantees that $2I - w^2 \neq 0$.

[40] Parts 3) and 11) of Lemma 0 guarantee that $2I - w^2 \neq 0$.

[41] $w \neq c_1(t)$ guarantees $I - w^2 - (2I - w^2)t \neq 0$.