Permutation Centralizer Algebras, Polynomial Invariants and supersymmetric states

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Representation theory and Physics, Leeds, July 2016

"Permutation centralizer algebras and multi-matrix invariants," Mattioli and Ramgoolam
arxiv:1601.06086, Phys. Rev. D.
"Quivers as Calculators : Counting, correlators and Riemann surfaces," J. Pasukonis, S. Ramgoolam
arxiv:1301.1980, JHEP, 2013
"Branes, anti-branes and Brauer algebras," Y. Kimura, S. Ramgoolam, arXiv:0709.2158, JHEP, 2007.
More complete references are in these papers.

Introduction : $\mathbb{C}(S_n)$

S_n the symmetric group of all permutations of $\{1, 2, \cdots, n\}$.

The group algebra $\mathbb{C}(S_n)$ spanned by formal linear combinations of S_n group elements.

$$a = \sum_{\sigma \in S_n} a_\sigma \, \sigma$$

Can be scaled, added, multiplied e.g.

Product

$$\begin{aligned} ab &= \sum_{\sigma \in S_n} a_\sigma \, \sigma \sum_{\tau \in S_n} b_\tau \, \tau \\ &= \sum_{\sigma, \tau} a_\sigma b_\tau \;\; \sigma\tau \end{aligned}$$

PCA Example 0 : The centre $\mathcal{Z}(\mathbb{C}(S_n))$

The subspace of elements which commute with everything.

Spanned by elements of the form

$$\bar{\sigma} = \sum_{\gamma \in S_n} \gamma \sigma \gamma^{-1}$$

One in each conjugacy class, e.g. in S_3

$$(1,2,3) + (1,3,2)$$
$$(1,2) + (1,3) + (2,3)$$
$$()$$

Dimension

The dimension of the centre $\mathcal{Z}(\mathbb{C}(S_n))$ is equal to the number of partitions of n, denoted $p(n)$.

$$3 = 3$$
$$3 = 2 + 1$$
$$3 = 1 + 1 + 1$$

$p(3) = 3$.

Fourier transform and Young diagrams

Another basis for the centre is given by Projectors, constructed from characters.
One for each irrep of S_n, i.e. one for each Young diagram R

$$P_R = \frac{d_R}{n!} \sum_{\sigma \in S_n} \chi^R(\sigma) \sigma$$

$$d_R = \text{ Dimension of } R$$

$$D^R(\sigma) : V_R \to V_R$$
$$\chi^R(\sigma) = \mathrm{tr}(\mathrm{D}^{\mathrm{R}}(\sigma))$$

$$P_R P_S = \delta_{RS} P_R$$

Example 1 : $\mathcal{A}(m,n)$

Consider the sub-algebra of $\mathbb{C}(S_{m+n})$ which commutes with $\mathbb{C}(S_m \times S_n)$. This is spanned by elements

$$\bar{\sigma} = \sum_{\gamma \in S_m \times S_n} \gamma \sigma \gamma^{-1}$$

This is a non-commutative associative, semi-simple algebra. Has a non-degenerate pairing.

- Dimension $p(m,n)$

- Fourier transform and a basis in terms of triples of Young diagrams: Triple $(R_1, R_2, R, \nu_1, \nu_2)$

- $R_1 \vdash m, R_2 \vdash n, R \vdash (m+n)$

$$1 \leq \nu \leq (g(R_1, R_2, R)$$

Applications in Invariant theory

Two matrices Z, Y of size N, with matrix entries z_{ij}, y_{ij}. We are interested in polynomial functions of these, which are invariant under

$$(Z, Y) \to (UZU^\dagger, UYU^\dagger)$$

U is a unitary matrix.

These are traces of matrix products, and products of traces

$$\mathrm{tr(ZZYY)}, \mathrm{tr(ZYZY)}$$

First fundamental theorem.

Applications in Matrix Integrals

We are interested in

$$\int dZ d\bar{Z} dY d\bar{Y} Y e^{-\mathrm{tr}ZZ^\dagger - \mathrm{tr}YY^\dagger} P(Z, Y) Q(Z^\dagger, Y^\dagger) = \langle PQ \rangle$$

where P, Q are gauge-invariant polynomials (invariant under the actions before).

The enumeration of the invariants $P(Z, Y)$ for degree m, n, when $m + n \leq N$, is related to $\mathcal{A}(m, n)$. For $m + n > N$, it is related to an N-dependent quotient of $\mathcal{A}(m, n)$, which we will call $\mathcal{A}_N(m, n)$.

$\mathcal{A}_N(m, n)$ also knows about the correlators of these gauge-invariant polynomials.

General definition : Permutation centralizer algebras

Start with an associative algebra $\mathcal{A}$ which contains the group algebra of a permutation group H. Consider the sub-algebra of $\mathcal{A}$ which commutes with $\mathbb{C}(H)$.

This is a Permutation Centralizer algebra.

Example 2 : $\mathcal{A} = B_N(m, n)$ - the walled Brauer algebra. The permutation sub-algebra is $\mathbb{C}(S_m \times S_n)$.

Example 3: $\mathcal{A} = \mathbb{C}(S_n) \otimes \mathbb{C}(S_n)$. The interesting sub-algebra is the centralizer of $\mathbb{C}(Diag(S_n))$.

Outline of Talk

- Properties of $\mathcal{A}(m, n)$: Relations to LR coeffients ; Quotient;
- $\mathcal{A}(m, n) \rightarrow$ Matrix invariants and Correlators.
- Physics applications - AdS/CFT. Quantum states. Enhanced symmetries and Charges.
- Charges and the structural question on $\mathcal{A}(m, n)$.
- Open problems and other examples.

Part 1 : Properties of $\mathcal{A}(m, n)$ - dimension

$$\sigma \in S_{m+n} \qquad , \qquad \gamma \in S_m \times S_n$$
$$\sigma \quad \sim \quad \gamma\sigma\gamma^{-1}$$

The number of equivalence classes under sub-group conjugation can be computed by using Burnside Lemma

$$p(m, n) = \frac{1}{m!n!} \sum_{\gamma \in S_m \times S_n} \sum_{\sigma \in S_{m+n}} \delta(\sigma\gamma\sigma^{-1}\gamma^{-1})$$

This leads to a generating function

$$\sum_{m,n} p(m, n) z^m y^n = \prod_{i=1}^{\infty} \frac{1}{(1 - z^i - y^i)}$$

Part 1 : $\mathcal{A}(m, n)$ - Dimension in terms of Young diagrams

The Burnside formula can be re-written in terms of a triple of Young diagrams.

$$
\begin{aligned}
&R_1 \vdash m \\
&R_2 \vdash n \\
&R \vdash m + n
\end{aligned}
$$

and $g(R_1, R_2, R)$ is the Littlewood-Richardson coefficient.

$$\sum_{R \vdash m+n} \sum_{R_1 \vdash m} \sum_{R_2 \vdash n} (g(R_1, R_2, R))^2$$

Part 1 : LR coefficients and reduction multiplicities

For some Young diagram R with $m+n$ boxes, we have an irrep V_R of S_{m+n}. The reduction to $S_m \times S_n$ produces

$$V_R = \bigoplus_{R_1, R_2} V_{R_1} \otimes V_{R_2} \otimes V_R^{R_1, R_2}$$

States $|R, I >$ in the V_R irrep can be expanded in terms of sub-group irreps

$$|R_1, i_1, R_2, i_2, \nu\rangle$$

The ν runs over the multiplicity space $V^R_{R_1, R_2}$.

$$Dim(V_R^{R_1, R_2}) = g(R_1, R_2, R)$$

Part 1 : Projector-like basis for $\mathcal{A}(m, n)$

In the case of $\mathcal{Z}(\mathbb{C}(S_n))$, we had $p(n)$ conjugacy classes and a projector basis.

$$P_R \propto \sum_{\sigma \in S_n} \chi^R(\sigma)\sigma$$

Now we have a

$$Q^{R_1,R_2,R}_{\nu_1,\nu_2} \propto \sum_{\sigma \in S_{m+n}} \chi^{R_1,R_2,R}_{\nu_1,\nu_2}(\sigma)\sigma$$

Part 1 : Explicit formula for quiver character

quiver character

$$\chi^{R}_{R_1,R_2,\nu_1,\nu_2}(\sigma)$$

can be written in terms of matrix elements of

$$D^{R}(\sigma) : V_R \to V_R$$

and overlaps (branching coefficients)

$$\langle R, I | R_1, i_1, R_2, i_2, \nu_1 \rangle$$

The formula involves a trace over states within irreps of the subgroup.

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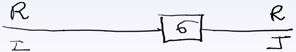
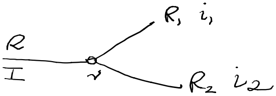
$D^{R}_{IJ}(\sigma)$:



$\langle R, I \mid R_1\, i_1\, R_2\, i_2\, \nu \rangle$



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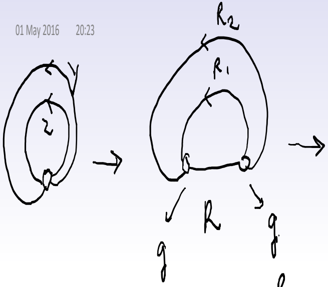

$$\sum_{R_1,R_2,R} \left(g(R_1,R_2,R) \right)^2$$

$$\chi^{R,R_1,R_2}_{v_1,v_2}(\sigma) =$$

Part 1 : Wedderburn-Artin for $\mathcal{A}(m,n)$

These projector-like elements have matrix-like multiplication properties

$$Q^{\vec{R}}_{\nu_1,\nu_2} Q^{\vec{S}}_{\mu_1,\mu_2} = \delta^{\vec{R},\vec{S}} \delta_{\nu_2,\mu_1} Q^{\vec{R}}_{\nu_1,\mu_2}$$

This is the Wedderburn-Artin decomposition of the $\mathcal{A}(m,n)$. Isomorphism between an associative algebra (with non-degenerate bilinear pairing) and a direct sum of matrix blocks.

Part 1 : Number of blocks

The dimension of the algebra is

$$\sum_{R_1 \vdash m} \sum_{R_2 \vdash n} \sum_{R \vdash m+n} (g(R_1, R_2, R))^2$$

The number of blocks is number of triples (R_1, R_2, R) with non-vanishing Littlewood-Richardson coefficients. Within each block, unit matrix is central in $\mathcal{A}(m, n)$.
These are projectors

$$P^{R_1, R_2, R} = \sum_{\nu} Q^{R, R_1, R_2}_{\nu, \nu}$$

Dimension of centre is the number of triples (R_1, R_2, R) with non-vanishing LR coeffs.

Part 1 : Dimension of Cartan

The WA-decomposition gives a maximally commuting sub-algebra (Cartan) : the span of

$$Q^{R_1, R_2, R}_{\nu, \nu}$$

The dimension of this Cartan is

$$\sum_{R_1 \vdash m} \sum_{R_2 \vdash n} \sum_{R \vdash m+n} g(R_1, R_2, R)$$

Part 1 : Finite N quotient.

For matrix theory and AdS/CFT applications, it will be useful to consider the quotient

$$\mathcal{A}_N(m,n)$$

defined by setting to zero all the Q's where the Young diagram R with $m+n$ boxes has no more than N rows.

Part 2 : Applications to Matrix Integrals and AdS/CFT

In 4D CFT, we have the operator-state correspondence of radial quantization.

Quantum states correspond to "local operators" at a point in 4D spacetime.

These local operators are gauge invariant polynomial functions of the "elementary fields."

In many CFTs of interest, the theory has a $U(N)$ gauge symmetry. The fields include complex matrices Z, Y which transform in the adjoint of the $U(N)$.

This includes $\mathcal{N} = 4$ SYM which is dual to $AdS_5 \times S^5$ string theory.

Invariant theory $\rightarrow$ gravitons and branes in 10 dimensions.

Correlators $\rightarrow$ combinatorics

We want to compute the 4D path integral -

$$\langle \mathcal{O}_a(Z,Y)(x_1)(\mathcal{O}_b(Z,Y)(x_2))^\dagger \rangle$$

For two-point functions of this sort, in the free-field limit, the dependence on x_1, x_2 is trivial. The non-trivial part of the problem is to find a good way to enumerate the gauge invariants and to express the dependence on the choice of a, b. This combinatoric problem can be formulated in a reduced zero-dimensional matrix model.

Matrix Invariants $\rightarrow$ Permutation equivalences

First step is the enumeration problem. Key observation is that, for fixed numbers of Z, Y, the gauge-invariants can be parametrized by permutations $\sigma \in S_{m+n}$

$$\begin{aligned} \mathcal{O}_\sigma(Z, Y) &= Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_n}_{i_{\sigma(m)}} Y^{i_{m+1}}_{i_{\sigma(n+1)}} \cdots Y^{i_{n+m}}_{i_{\sigma(m+n)}} \\ &= tr_{V^{\otimes m+n}}(Z^{\otimes m} \otimes Y^{\otimes n} \sigma) \end{aligned}$$

Exercise shows that

$$\mathcal{O}_{\gamma\sigma\gamma^{-1}} = \mathcal{O}_\sigma$$

for $\gamma \in S_m \otimes S_n$.

Path Integral $\implies$ matrices Y, Z gone

The two point functions are some functions of (σ_1, σ_2), computable using Wick's theorem, which are invariant under independent conjugations of σ_1, σ_2 by the sub-group. Explicit formulae can be written for

$$\langle \mathcal{O}_{\sigma_1}(Y,Z)(\mathcal{O}_{\sigma_2}(Y,Z))^\dagger \rangle$$

in terms of the product in $\mathcal{A}(m,n)$,

$$\sum_{\sigma_3 \in S_{m+n}} \sum_{\gamma \in S_m \times S_n} \delta(\sigma_1 \gamma \sigma_2 \gamma^{-1} \sigma_3) N^{C_{\sigma_3}}$$

Finite N counting from $U(N)$ group integrals

Finite N effects are very interesting in the phyics - related to giant gravitons ; also thermodnamics of the theory.
There is a $U(N)$ group integral formula (Sundborg, 2000) for the counting of the dimension of the space of operators. This can be manipulated to show that the formula is

$$\sum_{\substack{R \vdash m+n \\ l(R) \leq N}} \sum_{R_1 \vdash m} \sum_{R_2 \vdash n} (g(R_1, R_2, R))^2$$

Orthogonal Basis at finite N

We can form linear combinations of the permutation operators, labelled by the representation labels

$$\mathcal{O}^{R_1,R_2,R}_{\nu_1,\nu_2}(Z,Y) = tr_{V^{\otimes m+n}}\left(Z^{\otimes m}\otimes Y^{\otimes n} Q^{R_1,R_2,R}_{\nu_1,\nu_2}\right)$$

Theorem
The two point function of the representation-labelled operators is **diagonal.**

$$\langle \mathcal{O}^{\vec{R}}_{\mu_1,\mu_2}(Z,Y)(\mathcal{O}^{\vec{S}}_{\nu_1,\nu_2}(Z,Y))^\dagger\rangle = \delta^{\vec{R},\vec{S}}\delta_{\mu_1,\nu_1}\delta_{\mu_2,\nu_2} f_{\vec{R}}(N)$$

Bhattacharrya, Collins, de Mello Koch, 2008 ; Kimura, Ramgoolam, 2007 , Brown, Heslop, Ramgoolam, 2007

Orthogonal bases and Hermitian operators

This was in fact, one orthogonal basis, for 2-matrix invariants. Another is labelled by representations of the $U(2)$ which acts on the Z, Y pair. Yet another related to Brauer algebras (more later).

Orthogonal bases of quantum states are related to Hermitian operators with distinct eigenvalues. So what are the operators which distinguish these orthogonal states of the 2-matrix problem ?

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Enhanced symmetries of gauge theory and resolving the spectrum of local operators,

Kimura, Ramgoolam - Phys Rev D 2008

Enhanced symmetries, Casimirs, Charges

The two-point functions in the CFT define an inner product for these invariants.

The free field action

$$\int d^4x tr(\partial_\mu Z \partial_\mu Z^\dagger) + tr(\partial_\mu Y \partial_\mu Y^\dagger)$$

has symmetries

$$Y \to UY$$
$$Z \to VZ$$

where U, V are $U(N)$ group elements.

Noether charges for these symmetries $(E_{L,y})^i_j$ and $(E_{L,z})^i_j$ form $u(N) \times u(N)$ Lie algebra.

Casimirs built from $(E_z)^i_j$ measure the R_1 Young diagram label :

$$[(E_z)^i_j (E_z)^j_i, \mathcal{O}^{R_1,R_2,R}_{\nu_1,\nu_2}] = C_2(R_1) \mathcal{O}^{R_1,R_2,R}_{\nu_1,\nu_2}$$

These Casimirs - by Schur-Weyl duality - can be expressed in terms of central elements of S_n acting on the $Q^{R_1,R_2,R}_{\nu_1,\nu_2}$.

Casimirs built from $(E_y)^i_j$ measure R_2.

"Mixed Casimirs" such as

$$(E_{y,L})^i_j (E_{y,L})^j_k (E_{z,L})^k_i$$

are sensitive to the multiplicity label ν_1.

These left actions amount to the action of $\mathcal{A}(m, n)$ on itself from the left.

Physics question: Find a minimal complete set of Casimir charges which uniquely determine the representation labels $R_1, R_2, R, \nu_1, \nu_2$ of the quantum states.

This is a measure of the complexity of the state space of the 2-matrix quantum states.

Charges $\rightarrow$ structure of $\mathcal{A}(m,n)$

The physics question translates into some maths questions

1. What is a minimal set of generators for the centre $\mathcal{Z}(\mathbb{C}(S_n))$? Experiments show that at low n (around 12) the sums over transpositions suffice to generate. To go a bit higher we can use sums over (ij) and (ijk).

2. In $\mathcal{A}(m,n)$ we described a Cartan $\mathcal{M}(m,n)$ and a centre $\mathcal{Z}(\mathcal{A}(m,n))$. If we consider elements in $\mathcal{M}(m,n)$ with coefficients in $\mathcal{Z}$

$$\sum_i z_i m_i$$

what is the minimal dimension of a generating subspace ?

Exploiting the structure of $\mathcal{A}(m,n)$ for Matrix integrals.

Central elements in $\mathcal{A}(m,n)$, correspond to a subset of matrix invariants.

Their correlators can be computed using characters of S_m, S_n, S_{m+n}, without the need for branching coefficients etc.

Thus, for example, explicit formulae for

$$\langle tr(Z^m Y^n) tr((Z^m Y^n)^\dagger)\rangle$$

The Brauer example

$B_N(m,n)$ - centralizer of $U(N)$ acting on $V^{\otimes m} \otimes \bar{V}^{\otimes n}$.
subalgebra which commutes with $\mathbb{C}(S_m \times S_n)$.
Fourier basis

$$Q^{\gamma}_{\alpha,\beta,i,j}$$

γ is an irrep of Brauer, labelled by $(k, \gamma_+ \vdash m-k, \gamma_- \vdash n-k)$.
α is a rep of S_m , β is a rep of S_n. The indices i,j run over multiplicity of irrep (α, β) of $S_m \times S_n$ in γ.
Can be used to build a basis for matrix invariants for $Z, Z^{\dagger}$.

Branes, anti-branes and Brauer algebras, Kimura and Ramgoolam, 2007

$$Z \longrightarrow U Z U^{+}$$

$$Z^{+} \longrightarrow U Z^{+} U^{+}$$

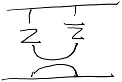
$$\mathrm{tr}(Z Z^{+}) = Z^{i_1}_{i_2} Z^{+i_2}_{i_1}$$

$$= Z^{i_1}_{i_{\sigma(1)}} Z^{+i_2}_{i_{\sigma(2)}}$$

$$= \mathrm{tr}_{V\otimes V}(Z \otimes Z^{+} \sigma)$$

$$\mathrm{tr}\, Z Z^{+} = Z^{i_1}_{i_2} \bar{Z}^{i_1}_{i_2}$$

$=$



$$= \mathrm{tr}_{V\otimes \bar{V}}(Z \otimes \bar{Z}\, b)$$

For $m + n < N$, we have

$$\sum_{R_1,R_2,R} (g(R_1, R_2, R))^2 = \sum_{\alpha,\beta,\gamma} (M^{\gamma}_{\alpha,\beta})^2$$

When $m + n > N$, we know how to count the matrix invariants, using a simple modification of the LHS :

$$l(R) \leq N$$

A simple cut-off on γ does not do the job. Non-semi-simplicity. How to do the Brauer counting of invariants for finite N ?