Representation theory of Symmetric Groups and Local operators in Gauge theory

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J. Pasukonis and S. Ramgoolam, http://arxiv.org/abs/1010.1683 ;
T.W. Brown, P. Heslop, S. Ramgoolam, http://arxiv.org/abs/0711.0176
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AdS-CFT

$\mathcal{N} = 4$ gauge theory

with $U(N)$ gauge group

$$\updownarrow$$

String theory $AdS_5 \times S^5$.

The gauge theory has $U(N)$ gauge group.

Hence the dynamical variables include a matrix-valued gauge field $A_\mu(\mathbf{x}, t)$. In addition, there are 6 bosonic hermitian matrix fields $\Phi_1(\mathbf{x}, t), \Phi_2(\mathbf{x}, t), \cdots, \Phi_6(\mathbf{x}, t)$.

There are also fermion fields λ^a which are spinors.

We are interested in the path integral

$$\mathcal{Z} = \int DAD\lambda D\Phi e^{-S(A,\Phi,\lambda)}$$

and correlators, such as

$$\langle tr\Phi_1^n(x) tr\Phi_1^n(y)\rangle \equiv \frac{1}{\mathcal{Z}} \int DAD\lambda D\Phi e^{-S(A,\Phi^a,\Lambda^a)} tr\Phi_1^n(x) tr\Phi_1^n(y)$$

The action depends on a coupling constant g_{YM}^2. Computations in the gauge theory can be done at $g_{YM}^2 = 0$ or small. Strong coupling of large $g_{YM}^2 N$ is easier on the dual AdS side.

$tr\,\Phi_1^n(y)$ is an example of a local operator. In conformal field theories, there is an operator-state correspondence, which allows us to associate a quantum state to a local operator.

This is done in the framework of radial quantization, where the radial displacement from a chosen point plays the role of time, and the scaling operator plays the role of Hamiltonian.

Will freely use **local operators or states**.

The theory is **superconformal**.

Conformality means that the bosonic symmetries include an operator Δ which implements the action of $x \to \lambda x$ on the fields.

The symmetries include 16 fermionic symmetric generators Q and sixteen conjugate generators S, such that

$$\{Q, S\} = \Delta + \cdots$$

The local-operators/states from representations of the superconformal algebra. These representations have $\Delta \geq 0$.

The lowest Δ (called lowest weight) states are annihilated by all the S symmetry generators.

The Q generate multiplets, which contain fields with different Poincare symmetries, e.g scalars and tensors $T_{\mu\nu}$ as well as fermions can belong to the same multiplet.

Special representations are generated by some operator $\mathcal{O}$ annihilated by some subset of the Q.

Operators annihlated by

8 of the Q are called half-BPS		
4	$\cdots\cdots$	quarter-BPS
2	$\cdots\cdots$	eighth-BPS

$\color{blue}{\text{Half-BPS lowest weight}}$ states are of the form $trX, trX^2, ..$ where X is a complex combination $(\Phi_1 + i\Phi_2)$, holomorphic traces in the matrix variable X along with products of these traces.

Their multiplicity and a subset of their correlators are unchanged from zero coupling $g_{YM}^2 N = 0$ to strong coupling of $g_{YM}^2 N$ large.

Quarter BPS lowest weight states at zero coupling are constructed from arbitrary holomorphic matrix traces of X, Y (two complex combinations).

$$\begin{aligned} X &= \Phi_1 + i\Phi_2 \\ Y &= \Phi_3 + i\Phi_4 \end{aligned}$$

At first order in the expansion in g^2_{YM}, or at one-loop, the multiplicity changes. Only those in the kernel of

$$\mathcal{H}_2 = tr[X, Y][\check{X}, \check{Y}]$$

are quarter BPS.

There is evidence that no further jump happens beyond one-loop. The multiplicity and an appropriate subset of correlators involving these quarter BPS operators remain unchanged.

A similar story of a jump at weak coupling holds for eighth BPS states. At zero coupling they are made from holomorphic traces of X, Y, Z.

For AdS/CFT, it is important to know this spectrum of states precisely. These states include gravitons, which arise from Kaluza-Klein reduction on the S^5, as well as large rotating branes and deformations of the $AdS_5 \times S^5$ geometries.

Symmetric groups (or permutation groups) have proved very useful in organising these states : providing a basis that works at finite N, and diagonalises the CFT-inner product on the states.

OUTLINE

1. The half BPS case. A basis for holomorphic traces in one matrix which diagonalizes the inner product. Labelled by Young diagrams
2. Quarter and eighth BPS in the limit $g_{YM}^2 = 0$. A basis in terms of Clebsch-Gordan coefficients of symmetric groups
3. Quarter and eighth BPS at weak coupling interacting. A matrix $\mathcal{PGP}$.
4. Why symmetric groups ? Schur-Weyl duality + enhanced Noether symmetries at zero coupling
5. Some comments on the dual AdS physics of these states.

HALF-BPS case

In this case, all the relevant states are obtained from gauge invariant polynomials in one complex matrix

$$
\begin{aligned}
n = 1: &\quad \mathrm{tr}(\mathrm{X}) \\
n = 2: &\quad \mathrm{trX}^2\ ,\ (\mathrm{trX})^2 \\
n = 3: &\quad \mathrm{trX}^3, \mathrm{trX}^2\mathrm{trX}, (\mathrm{trX})(\mathrm{trX})
\end{aligned}
$$

The counting of these states at degree *n* is given by the number of partitions of *n*.

$$
\begin{aligned}
1 &= 1 \\
2 &= 2 \\
  &= 1 + 1 \\
3 &= 3 \\
  &= 2 + 1 \\
  &= 1 + 1 + 1
\end{aligned}
$$

The number $p(n)$ is the number of conjugacy classes of S_n – the symmetric group of permutations of n objects.

$$\sigma \sim \alpha\sigma\alpha^{-1}$$

Conjugacy classes are specified by cycle structures of the permutation

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \to (12)(3)$$
$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \to (123)$$

In general the cycle structure of $\sigma \in S_n$ can be written as

$$1^{p_1} 2^{p_2} \cdots$$
$$n = p_1 + 2p_2 + \cdots$$

The matrix X, of size N can be viewed as

$$X : V \rightarrow V$$

where V is N-dimensional vector space.

$$X \otimes X \cdots \otimes X : V^{\otimes n} \rightarrow V^{\otimes n}$$

Action of permutations on $V^{\otimes n}$

$$\sigma(e_{i_1} \otimes e_{i_2} \cdots e_{i_n}) = e_{i_{\sigma(1)}} \otimes e_{i_{\sigma(2)}} \cdots e_{i_{\sigma(n)}}$$

The general polynomial can be written as

$$tr_{V^{\otimes n}}(X^{\otimes n}\sigma)$$

or more explicitly

$$X^{i_1}_{i_{\sigma(1)}} X^{i_2}_{i_{\sigma(2)}} \cdots X^{i_n}_{i_{\sigma(n)}}$$

These polynomials in X are independent under conjugation $\sigma \rightarrow \gamma\sigma\gamma^{-1}$

For the $n = 2$ case,

$$\begin{aligned}(trX)^2 &= X^{i_1}_{i_1} X^{i_2}_{i_2} = X^{i_1}_{i_{\sigma(1)}} X^{i_2}_{i_{\sigma(2)}} \text{ with } \sigma = (1)(2) \\ (trX^2) &= X^{i_1}_{i_2} X^{i_2}_{i_1} = X^{i_1}_{i_{\sigma(1)}} X^{i_2}_{i_{\sigma(2)}} \text{ with } \sigma = (12)\end{aligned}$$

This is a basis

$$\mathcal{O}_{[\sigma]}(X) = tr_{V^{\otimes n}}(X^{\otimes n}\sigma)$$

in the space of gauge-invariant polynomials as long as $n \leq N$. It is overcomplete for $n > N$.

Physically we want to compute the two-point function

$$\langle \mathcal{O}_{[\sigma_1]}(X)\mathcal{O}_{[\sigma_2]}(X^\dagger)\rangle$$

which define an inner product on the space of states corresponding to local operators $\mathcal{O}_{[\sigma_1]}$

The basic 2-point function is

$$\langle X^i_j (X^\dagger)^k_l \rangle = \delta^i_l \delta^k_j$$

At $g_{YM}^2 = 0$, we can use Wick's theorem to get

$$< XXXX^\dagger X^\dagger X^\dagger X^\dagger >= \sum < XX^\dagger >< XX^\dagger >< XX^\dagger >< XX^\dagger >$$

The sum is over different pairings of the X and $X^\dagger$. There are 4! in this case, in general n! with n copies of $X, X^\dagger$. These sums can also be parametrized by permutations in S_n .

The two-point function is complicated in the multi-trace basis. A better basis – good at finite N and diagonalises the 2-point function – is

$$\mathcal{O}_R(X) = \sum_\sigma \chi_R(\sigma)\mathcal{O}_\sigma(X)$$

It diagonalises

$$\langle \mathcal{O}_R(X)\mathcal{O}_S(X^\dagger)\rangle = f_R \delta_{R,S}$$

where R is a Young diagram of S_n.

For the Young diagram which has one row of two boxes (symmetric)

$$\chi_{[2]}(X) = \frac{1}{2}(trX^2 + trXtrX)$$

For the one with one box in each of two rows (antisymmetric)

$$\chi_{[1,1]}(X) = \frac{1}{2}(trX^2 - trXtrX)$$

There is a nice representation theory answer for these 2-point functions. In this basis, 3-point functions are simply related to Littlewood-Richardson coefficients for composing Young diagrams.

Permutations – more precisely conjugacy classes – appear in classifying the operators.

$$tr_{V^{\otimes n}}(X^{\otimes n}\sigma) = tr_{V^{\otimes n}}(X^{\otimes n}\gamma\sigma\gamma^{-1})$$

Wick contractions parametrized by permutations. So expect group theory.

Conjugacy classes and irreducible representations equal in number and paired by character $\chi_R(\sigma)$.

Eighth-BPS case : Free limit.

The eighth-BPS case. Gauge-invariant polynomials in three complex matrices X_1, X_2, X_3.

Now consider

$$X_{a_1} \otimes X_{a_2} \cdots X_{a_n}$$

and

$$\mathcal{O}_{\vec{a}}(\alpha) = \textbf{tr}\,(X_{a_1} \otimes X_{a_2} \cdots \otimes X_{a_n} \alpha)$$

Again for a good diagonal basis – we need some representation theory, In this case local operators not invariant under conjugation. But Wick contractions (hence permutations) still govern the correlator. So expect group theory – beyond characters.

General group elements related to representations

$$n! = \sum_R d_R^2$$

Fourier transform on group relates permutations σ to irreducible representation labels R, i, j by

$$D^R_{ij}(\sigma)$$

Expect these more refined representation theory data to appear in finding a diagonal basis for 2point functions.

There is $U(3)$ transformation group where X_1, X_2, X_3 form a 3-dimensional representation. Can work with a $U(M)$ version with $M = 3$ here.

$$V_M^{\otimes n} \equiv \bigoplus_{\Lambda \vdash n} V_\Lambda^{U(M)} \otimes V_\Lambda^{(S_n)}$$

Schur-Weyl duality.

There exists a unitary transformation

$$C^{a_1,\cdots,a_n}_{\Lambda,M_\Lambda,m_\Lambda} \langle a_1, a_2, \cdots, a_n | \Lambda, M_\Lambda, m_\Lambda \rangle$$

This is a type of generalized Clebsch-Gordan coefficient.

Another decomposition is

$$V_R^{(S_n)} \otimes V_R^{(S_n)} = \bigoplus_T V_T^{(S_n)} \otimes V_{R,R}^T$$

Tensor product of two irreps of S_n. There is a diagonal action of S_n. Decomposing into irreps under this diagonal action, there is a direct sum of irreps.

The space $V_{R,R}^T$ is a multiplicity space for the irrep T.

There are now orthogonal Clebsch-Gordan coefficients

$$S^{R\ R\ T;\tau}_{i\ j\ k} = \langle R, i; R, j | T, k, \tau \rangle$$

where τ runs over the multiplicity space.

A diagonal basis for the 2-point functions

$$\mathcal{O}_{\Lambda,M_\Lambda,R,\tau} = \frac{\sqrt{d_R}}{n!}\sum_{\alpha,\vec{a}} S^{R\ R\ \Lambda,\ \tau}_{i\ j\ m} D^R_{ij}(\alpha) C^{\vec{a}}_{\Lambda,M_\Lambda,m} \mathcal{O}_{\vec{a},\alpha}$$

The finite N constraint is that R has no more than n rows.

Eighth-BPS : Interacting theory

Here again, we want the two-point function. The correct BPS states are now in the Kernel of a 1-loop dilatation operator

$$H = \mathrm{tr}[X_1, X_2][\check{X}_1, \check{X}_2]$$

The problem is to find a basis for this Kernel and get a formula for the 2-point function for the Kernel states.

Again we do this by using symmetric groups.
Any permutation $\alpha \in S_n$ with p_1 one-cycles, p_2 two-cycles, etc.

$$\overbrace{(.)(.)\cdots(.)}^{p_1}\quad\overbrace{(..)(..)\cdots(..)}^{p_2}\cdots$$

is associated with a subgroup $C(\alpha)$ of γ such that $\gamma\alpha\gamma^{-1} = \alpha$.
This is isomorphic to

$$S_{p_1} \times (S_{p_2} \ltimes Z_2^{p_2}) \times (S_{p_3} \ltimes Z_3^{p_3}) \cdots$$

where $p_1, p_2, \cdots$ are the numbers of $1, 2, \cdots$ cycles of α.

There is another similar subgroup $S(\alpha)$ where the Z_{p_i} are replaced by S_{p_i}

Consider an element in the group algebra of S_n

$$\mathbb{P}_\alpha = \frac{1}{|S(\alpha)|} \sum_{\sigma \in S(\alpha) \subset S_n} \sigma$$

This is a projector

$$\mathbb{P}_\alpha \mathbb{P}_\alpha = \mathbb{P}_\alpha$$

$C(\alpha)$ is the symmetry group of $\mathbf{tr}((X_{a_1} \otimes \cdots \otimes X_{a_n})\alpha)$
$S(\alpha)$ is the symmetry group of symmetrized traces.

The element $\mathbb{P}_\alpha$ being in $\mathbb{C}(S_n)$ can be expanded

$$\mathbb{P}_\alpha = \sum_\beta p_{\beta,\alpha} \beta$$

The numbers $p_{[\alpha],[\beta]}$ are interesting generalizations of "sorting numbers" which appear in combinatorics.

Define the matrix $(\mathcal{P})^{\Lambda,R_1,\tau_1}_{\Lambda,R_2,\tau_2}$ using $\mathbb{P}_\alpha$ and representation theory data

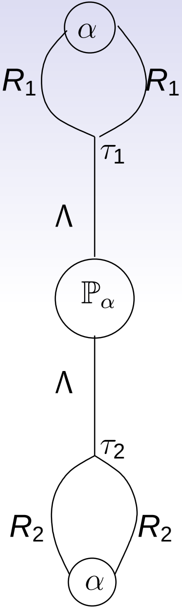
$$(\mathcal{P})^{\Lambda,R_1,\tau_1}_{\Lambda,R_2,\tau_2} = \frac{\sqrt{d_{R_1} d_{R_2}}}{n!} \sum_{\alpha\in S_n} D^{\Lambda}_{ij}(\mathbb{P}_\alpha)$$

$$D^{R_1}_{k_1 l_1}(\alpha) S^{R_1\;R_1\;\Lambda,\;\tau_1}_{k_1\;l_1\;i} D^{R_2}_{k_2 l_2}(\alpha) S^{R_2\;R_2\;\Lambda,\;\tau_2}_{k_2\;l_2\;j}$$

Define the matrix $(\mathcal{P})^{\Lambda,R_1,\tau_1}_{\Lambda,R_2,\tau_2}$ using $\mathbb{P}_\alpha$ and representation theory data

$$(\mathcal{P})^{\Lambda,R_1,\tau_1}_{\Lambda,R_2,\tau_2} = \frac{\sqrt{d_{R_1} d_{R_2}}}{n!} \sum_{\alpha\in S_n} D^{\Lambda}_{ij}(\mathbb{P}_\alpha)$$

$$D^{R_1}_{k_1 l_1}(\alpha) S^{R_1\ R_1\ \Lambda,\ \tau_1}_{k_1\ l_1\ i} D^{R_2}_{k_2 l_2}(\alpha) S^{R_2\ R_2\ \Lambda,\ \tau_2}_{k_2\ l_2\ j}$$


α
R1
R1
τ1
∧
ℙα
∧
τ2
R2
R2
α

$\mathcal{P}$ is a projector on the Hilbert space of the free theory.
Another matrix is defined as

$$(\mathcal{G})^{\Lambda,R_1,\tau_1}_{\Lambda,R_2,\tau_2} = \frac{d_R}{DimR}\delta^{R_1}_{R_2}\delta^{\tau_1}_{\tau_2}$$

The 2-point function of the BPS states in the interacting theory is given by the matrix

$$\mathcal{P}\mathcal{G}\mathcal{P}$$

For the half-BPS case, we knew the eigenvalues and eigenvectors of the matrix of 2-point functions. The e-vecs were labelled by R – Young diagrams. The e-vals were group theoretic f_R.

Here the matrix of 2-point functions is $\mathcal{PGP}$.

Would like to know more about eigenvalues and eigenvectors of this matrix – an entirely group theoretic quantity.

Why symmetric groups ?

Not a symmetry of the lagrangian.

They arise by Schur-Weyl duality. Back to the half BPS sector

Relavant part of the action is

$$S = \int d^4x \partial^\mu X \partial_\mu X^\dagger$$

The gauge symmetry

$$X \to UXU^\dagger$$

The traces are **invariant** under this $U(N)$.

The $\chi_R(X)$ are holomorphic continuations of $U(N)$ charcaters. So they ought to be related to some $U(N)$.

There is an enhanced global symmetry at zero coupling

$$X \to UXV^{\dagger}$$

The extra $U(N)$ explains the $U(N)$ nature of $\chi_R(X)$. Casimirs of the left $U(N)$ – constructed by Noether procedure – act diagonally on the $\chi_R(X)$.

For the case of 2-matrices we described a basis with manifest quantum numbers for $U(2)$.

Another basis uses the symmetry breaking $S_{m+n} \to S_m \times S_n$ arising when we consider

$$tr_{V^{\otimes(m+n)}}(X^{\otimes m} \otimes (X^{\dagger})^{\otimes n} \alpha)$$

This is the the restricted Schur basis. See papers of Robert de Mello Koch et. al.

Yet another basis uses a Brauer algebra $B(m, n)$ instead of S_{m+n} which also admits the $S_m \times S_n$ subgroup.

Existence of all these different bases can be understood by appropriate Casimir of enhanced symmetries.

See “Enhanced symmetries of gauge theory and resolving the spectrum of local operators,” Kimura and Ramgoolam, http://arxiv.org/abs/0807.3696

Back to ADS/CFT.

The connection between the Young diagram classification of half-BPS and spacetime in AdS is well understood : Giant gravitons and LLM geometries.

There are moduli spaces of eighth BPS giant graviton solutions – holomorphic surfaces in $\mathbb{C}P^3$. (Mikhailov)

The precise map between the local operators and these geometries is not understood. The results outlined here should help make contact with the quantum properties of these giant gravitons.

Longer term goal would be an analogous treatment of sixteenth BPS states, which are related to black holes having finite horizon area. Basic puzzles on counting and entropy are outstanding here.